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ON THE EXISTENCE OF PICARD OPERATORS IN SOME CLASSES OF GENERALIZED METRIC SPACES

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Introduction

1. History and reasons for choosing the topic

In 1922, S. Banach proved a famous theorem that is now commonly known as the "Banach contraction mapping principle".

Theorem 1. *Let (X, ρ) be a complete metric space. Suppose the mapping $T : X \to X$ satifies the contractive condition*

$$\rho(Ta, Tb) \leqslant r\rho(a, b) \text{ for all } a, b \in X, \tag{0.1}$$

where $r \in (0, 1]$ is a constant. Then T has a unique fixed point $\bar{a} \in X$ and for any $a \in X$, iterative $\{T^n a\}$ converges to $\bar{a}$.

This theorem of S. Banach is considered very important, it gave out a new research trend in the development of fixed point theory, which is the metric fixed point theory. Recently, the studied metric fixed point has attracted the attention of many domestic and foreign authors, obtainied many important results and had many applications in different fields of Mathematics such as studying the existence of solutions of differential equations, systems of linear equations, and integral equations,....

The Banach contraction mapping principle gives us a sufficient condition that a mapping from the complete metric space X into itself has a unique fixed point. Many authors have sought to develop the Banach contraction principle with different contraction conditions and in different space classes. For example, M. Edelstein in 1962, E. Rakotch in 1962, A. Meir and E. Keeler in 1969, Lj. B. Ćirić in 1974, A. C. M. Ran et al. in 2004, M. Berinde and V. Berinde in 2007, G. L. Huang and X. Zhang in 2007, T. Suzuki in 2007, Sh. Rezapour and R. Hamlbarani in 2008, T. Suzuki in 2009, W. S. Du in 2010, D. Wardowski in 2012, R. Pant in 2016, S.-i. Ri in 2016 and others. When studying the fixed point of mapping, in 1983, I. A. Rus introduced the concept of Picard and weakly Picard operators in metric space. Generalizing that concept for the class of topological spaces, we have the following definition:

Definition 1. Let X be a topological space. A mapping $T : X \to X$ is called *weakly Picard operator* if T has a fixed point and for each $a \in X$, sequence $\{T^n a\}$ converges to a fixed point of T. If T is a weakly Picard operator and has a unique fixed point, then T is called *Picard operator*.

From the above definition, we see that the Picard and weakly Picard operators are closely related to the fixed point of the mapping, for example, the Banach contraction mapping principle is a Picard operator on the complete metric space. In the results of I. A. Rus, M. Berinde and V. Berinde and others, the authors have studied some properties of the Picard and weakly Picard operators relating to the set of fixed points of single-valued and multivalued mappings. In this thesis, we focus on studying the existence of Picard operators associated with the contractive conditions. Recent research results in this trend are divided into three main issues:

1. Construct some sufficient conditions with different contraction conditions so that a mapping is a Picard or weakly Picard operators on the class of metric spaces.

2. Construct some structured spaces that are extended from the class of metric spaces (we often call them generalized metric spaces) and construct some sufficient conditions relating to the contraction conditions, so that a mapping is a Picard or weakly Picard operators on these space classes.

3. Study the different applications of the Picard and the weakly Picard operators.

According to the first trend, the authors extended the Banach contraction mapping condition and construct some new contraction conditions so that a mapping is a Picard or weakly Picard operators. In 1962, M. Edelstein established sufficient condition for a mapping to be the Picard operator for compact metric spaces: *Let* (X, ρ) *be a compact metric space and let* $T : X \to X$ *be a mapping. Suppose* $\rho(Ta, Tb) < \rho(a, b)$ *for all* $a, b \in X, a \neq b$*, then* T *is a Picard operator.* Following the result of M. Edelstein, many authors have developed of the Banach contraction mapping principle in the metric space by replacing the constant r in the condition (0.1) by a parameter, a function or restricting the condition (0.1) to be true only for some elements $a, b \in X$. For example, A. Meir and E. Keeler establish a sufficient condition for the mapping to be the Picard operator in the complete metric space (X, ρ) with the condition: *For each* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $\varepsilon \leqslant \rho(a, b) < \varepsilon + \delta$ *implies* $\rho(Ta, Tb) < \varepsilon$ *for all* $a, b \in X$; in 2016, S.-i. Ri replaces the constant contraction by a parameter function and obtains: *Let* (X, ρ) *be a complete metric space and* $T : X \to X$ *be a mapping. Suppose there exists function* $\varphi : (0, +\infty) \to (0, +\infty)$ *satisfying* $\varphi(t) < t, \limsup_{s \to t^+} \varphi(s) < t$ *for any* $t > 0$ *and* $\rho(Ta, Tb) \leqslant \varphi(\rho(a, b))$ *for all* $a, b \in X$*. Then,* T *is a Picard operator.* And more in 2007, by using a nonincreasing parameter function, T. Suzuki obtained the following result.

Theorem 2. *Let* (X, ρ) *be a complete metric space and mapping* T *from* X *to itself. Non-incrementing function* $\varphi : [0, 1) \to (\frac{1}{2}, 1]$ *is defined by*

$$\varphi(r) = \begin{cases} 1 & \textit{if } 0 \leqslant r \leqslant \frac{\sqrt{5}-1}{2}, \\ (1-r)r^{-2} & \textit{if } \frac{\sqrt{5}-1}{2} \leqslant r \leqslant 2^{-\frac{1}{2}}, \\ (1+r)r^{-1} & \textit{if } 2^{-\frac{1}{2}} \leqslant r < 1. \end{cases}$$

Suppose there exists $r \in [0, 1)$ *such that*

$$\varphi(r)\rho(a, Ta) \leqslant \rho(a, b) \ \textit{ implies } \ \rho(Ta, Tb) \leqslant r\rho(a, b),$$

for all $a, b \in X$*. Then,* T *is a Picard operator.*

The construction of the new contraction conditions, which are different from Banach contraction mapping condition, also attracted many authors. For example J. Górnicki, G. E. Hardy and T. D. Rogers, S. Reich .... In this dissertation, we are interested in the Kannan contraction mapping class. In 1968, R. Kannan proved the following fixed point theorem.

Theorem 3. *Let* (X, ρ) *be a complete metric space and mapping* T *from* X *to itself. Suppose there exists* $r \in [0, \frac{1}{2})$ *such that*

$$\rho(Ta, Tb) \leqslant r\big(\rho(a, Ta) + \rho(b, Tb)\big) \textit{ for all } a, b \in X. \tag{0.2}$$

Then, T *is a Picard operator.*

A mapping that satisfies the assumption of Theorem 3 is called a Kannan mapping. In 1968, R. Kannan showed a specific case of discontinuous Kannan mapping, which is a different from

Banach contraction mappings. Another important application of the Kannan mapping is to be able to describe the completeness of the metric space in terms of the fixed point property of the mapping. This was proved by P. V. Subramanyam in 1975, that is, *"The metric space (X,ρ) is complete if and only if every Kannan mapping has a unique fixed point* ". Note that Banach's contraction mappings does not have this property. Therefore, the class of mapping in Theorem 3 immediately attracted the interest of many authors, such as L. S. Dube and S. P. Singh, J. Górnicki, G. Hiranmoy et al. and others.

Define

$$\mathcal{S} = \{f : (0,\infty) \to [0,\frac{1}{2}) : f(t_n) \to \frac{1}{2} \text{ implies } t_n \to 0 \text{ as } n \to \infty\},$$
$$\mathcal{H} = \{\varphi : (0,\infty) \to [0,\frac{1}{3}) : \varphi(t_n) \to \frac{1}{3} \text{ implies } t_n \to 0 \text{ as } n \to \infty\}.$$

By using the above control function, in 2018, J. Górnicki obtained the following results:

Theorem 4. *Let (X,ρ) be a complete metric space and $T : X \to X$ be a mapping. Suppose there exists $f \in \mathcal{S}$ such that for each $a, b \in X$ whit $a \neq b$,*

$$\rho(Ta,Tb) \leq f(\rho(a,b))\big(\rho(a,Ta)+\rho(b,Tb)\big).$$

Then, T is a Picard operator.

Theorem 5. *Let (X,ρ) be a complete metric space and $T : X \to X$ be a mapping. Suppose there exists $\varphi \in \mathcal{H}$ such that for each $a, b \in X$ whit $a \neq b$,*

$$\rho(Ta,Tb) \leq \varphi(\rho(a,b))\big(\rho(a,Ta)+\rho(b,Tb)+\rho(a,b)\big).$$

Then, T is a Picard operator.

We note that the above results of J. Górnicki is an extension of Theorem 3 of R. Kannan. In 2014, with the idea of combining Banach and Kannan co-conditions, K. Farshid et al. established a sufficient condition for the mapping to be a weakly Picard operator as following.

Theorem 6. *Let (X,ρ) be a complete metric space and let $T : X \to X$ be a mapping. Suppose that T satisfies the following condition*

$$\rho(Ta,Tb) \leqslant M(a,b)\rho(a,b) \textit{ for all } a,b \in X,$$

where

$$M(a,b) = \frac{\rho(a,Tb)+\rho(b,Ta)}{\rho(a,Ta)+\rho(b,Tb)+1}.$$

Then

(1) *T is weakly Picard operator;*

(2) *If $\bar{a}, \bar{b} \in X$ are two distinct fixed points of T, then*

$$\rho(\bar{a},\bar{b}) \geqslant \frac{1}{2}.$$

And in 2017, Y. U. Gaba established the similar result as K. Farshid for the case of the $G-$metric spaces. Furthermore recently, there are some authors studying multivalued Kannan mapping. Let (X,D,K) be a strong $b-$metric space, denoted $CB(X)$ be the collection of all non-empty, closed, and bounded subsets of X. The function H defining by

$$H(A,B) := \max\{\sup_{a\in B} d(a,A), \sup_{a\in A} d(a,B)\},$$

where $A, B \in CB(X)$ and $d(a, A) := \inf_{b \in A} D(a, b)$, is called *metric Hausdorff* on $CB(X)$ induced by strong metric D. In 1991, I. A. Rus introduced the concept of multivalued weakly Picard operators. Generalizing that concept for the class of strong $b-$metric spaces, we have the following definition:

Definition 2. Let (X, D, K) be a strong $b-$metric space. A mapping $T : X \to CB(X)$ is called *multivalued weakly Picard operator* if T has a fixed point, i.e. there exists $\bar{a} \in X$ such that $\bar{a} \in T\bar{a}$ and for each $a \in X$, for each $b \in Ta$, there exists a sequence $\{a_n\}$ satisfying:

(i) $a_0 = a, a_1 = b$;

(ii) $a_{n+1} \in Ta_n$ for all $n = 0, 1, \dots$;

(iii) the sequence $\{a_n\}$ converges to the fixed point of X.

If T is a multivalued weakly Picard operator and has a unique fixed point, then T is called *multivalued Picard operator*.

In 1970, L. S. Dube and S. P. Singh proved a type of Theorem 3 for the multivalued mapping case as following:

Theorem 7. *Let (X, ρ) be a complete metric space and let be a mapping continuous multivalued $G : X \to CB(X)$. Assume there exists $s \in [0, \frac{1}{2})$ such that*

$$H(Ta, Tb) \le s\big(d(a, Ta) + d(b, Tb)\big) \text{ for all } a, b \in X.$$

Then, T is a multivalued weakly Picard operator.

In addition to the result of L. S. Dube and S. P. Singh, there are some results of authors on the existence of multivalued weakly Picard operators. For example, M. Berinde and V. Berinde, A. Felhi, I. A. Rus et al. and others.

In the second research direction, the authors focus on construction and studying the properties of some spaces similar structures or extending from the metric space and establish some sufficient conditions for a mapping to be the Picard or weakly Picard operators on these spaces. Some examples of constructed spaces are $b-$metric space, $G-$metric space, $2-$metric space, strong $b-$metric space, and geometric space and some other spaces. In particular, in 2007, L. G. Huang and X. Zhang introduced the cone metric space by replacing the set of real positive numbers $\mathbb{R}$ in the usual metric definition by a directional cone in the Banach space. The authors have established some sufficient conditions for the mapping to be the Picard operator under the normal cone assumption and this results are extensions of Theorem 1 and Theorem 3. In 2008, Sh. Rezapour and R. Hamlbarani re-proved the result of L. G. Huang and X. Zhang without the normality of cones. In 2014, while studying the fixed point theorem in the strong $b-$metric space, W. Kirk and N. Shahzad posed the question: "Is every space $b-$metric strong X dense in the complete strong $b-$metric space X'?" If so, then every contraction mapping $T : X \to X$ extends to a contraction mapping $T' : X' \to X'$ for which T' has a unique fixed point in the complete strong $b-$metric space. The above question was answered by T. V. An and N. V. Dung in 2016.

Theorem 8. *Let (X, D, K) be a strong $b-$metric space. Then*

(i) *(X, D, K) has a completion;*

(ii) *The completion of (X, D, K) is unique in the sense that if (X_1^*, D_1^*, K_1) and (X_2^*, D_2^*, K_2) are two completions of (X, D, K) then there is a bijective isometry $\phi : X_1^* \to X_2^*$ which restricts to the identity on X.*

In the direction of applied research of Picard and weakly Picard operators. The authors have found profound applications of the fixed point theorems to generalized contraction mappings on structured spaces of the metric space type to different areas of Mathematics. Some results can be mentioned such as those of E. Berstovanská, V. Muresan, I. M. Oluru, I. A. Rus, J. Wang et al.... Hence, the continued development and study of generalized metric spaces, and the topological properties of these spaces, is very necessary.

The choice of thesis topic: **"On the existence of Picard operator in some classes of generalized metric spaces"** our aim is to enrich the research results on the properties of metric spaces, generalized metric and the sufficient conditions for the mapping is the Picard and weakly Picard operators on these spatial classes.

2. Purpose and research object

- **Research purpose**

The first purpose: Establish some sufficient conditions for the mapping to be the weakly Picard operators on the complete metric spaces.

Second purpose: Establish some sufficient conditions for the mapping to be the Picard and weakly Picard operators on the strong $b-$metric spaces.

Third purpose: Construct strong b-TVS cone metric space and study some properties of this space, in particular, establish some sufficient conditions for the mapping to be the Picard operators and prove the completion principle in this space.

- **Research object**

In this dissertation, we focus study:

1. The metric space, the strong $b-$metric spaces, the strong b-TVS cone metric spaces.
2. Picard and weakly Picard operators.

3. Overview of the dissertation

With the above purposes, in this dissertation we have obtained some main results as follows:

(1) *For the first purpose:* Based on the idea of Theorem 2 and Theorem 6, we have established a number of new results about the sufficient condition for a mapping in the complete metric space to be the weakly Picard operators as follows:

Theorem 1.1.1. *Let (X, ρ) be a complete metric space and $T : X \to X$ be a mapping. Suppose there exists $\alpha > 0$ such that*

$$\frac{1}{2}\rho(a, Ta) \leqslant \rho(a, b) \text{ implies } \rho(Ta, Tb) \leqslant M(a, b, \alpha)\rho(a, b),$$

for all $a, b \in X$, where

$$M(a, b, \alpha) = \frac{\rho(a, Tb) + \rho(b, Ta) + \rho(a, b)}{2\rho(a, Ta) + \rho(b, Tb) + \alpha}.$$

Then

(1) *T is weakly Picard operator;*

(2) *If $\bar{a}, \bar{b} \in X$ are two distinct fixed points of T, then*

$$\rho(\bar{a}, \bar{b}) \geqslant \frac{\alpha}{3}.$$

Our Theorem 1.1.1 is a fixed point theorem of the mapping developed from the Banach co-condition combined with the Kannan co-condition. The Example 1.1.2 in the thesis shows that the mapping class in Theorem 1.1.1 and the mapping class in Theorem 6 are independent. Using the Hausdorff distance, we prove a type of the Theorem 1.1.1 for the multivalued mapping.

Theorem 1.2.4. *Let (X, ρ) be a complete metric space and $T : X \to CB(X)$ be a multivalued mapping. Suppose there exists $\alpha > 0$ such that*

$$\frac{1}{2}d(a, Ta) \leqslant \rho(a, b) \text{ implies } H(Ta, Tb) \leqslant P(a, b, \alpha)\rho(a, b),$$

for all $a, b \in X$, where

$$P(a, b, \alpha) = \frac{d(a, Tb) + d(b, Ta) + \rho(a, b)}{2\delta(a, Ta) + \delta(b, Tb) + \alpha}, \quad \delta(a, A) := \sup_{b \in A} \rho(a, b).$$

Then

(1) *T is multivalued weakly Picard operator;*

(2) *If $\bar{a}, \bar{b} \in X$ are two fixed points of T then*

$$\rho^2(\bar{a}, \bar{b}) \geqslant \frac{\alpha}{3}H(T\bar{a}, T\bar{b}).$$

(2) *For the second purpose:* Using control functions, in 2021 we prove a result type of J. Górnicki for strong $b-$metric spaces.

Theorem 2.1.4. *Let (X, D, K) be a complete strong $b-$metric space and mapping T from X into itself. Suppose that there exists a function $f \in \mathcal{S}$ such that for all $a, b \in X, a \neq b$,*

$$D(Ta, Tb) \leqslant f(D(a, b))\big(D(a, Ta) + D(b, Tb)\big).$$

Then, T is a Picard operator.

Theorem 2.1.6. *Let (X, D, K) be a complete strong $b-$metric space and mapping T from X into itself. Suppose that there exists a function $\varphi \in \mathcal{H}$ such that for all $a, b \in X, a \neq b$,*

$$D(Ta, Tb) \leqslant \varphi(D(a, b))\big(D(a, Ta) + D(b, Tb) + D(a, b)\big).$$

Then, T is a Picard operator.

Clearly, when $K = 1$, Theorem 2.1.4 returns Theorem 4 and Theorem 2.1.6 returns Theorem 5. Furthermore, Example 2.1.5 and Example 2.1.7 show that the mapping class satisfying our theorems is an extension of the mapping class in Theorem 4 and Theorem 5.

Next, from the contraction condition of Theorem 2 and Theorem 3, we give the concept *Kannan-Suzuki mapping* in Definition 3 and established sufficient condition for mapping is Picard operators.

Definition 3. Let (X, D, K) be a strong $b-$metric space. We say that $T : X \to X$ is *Kannan-Suzuki mapping* if there exists $s \in [0, \frac{1}{2})$ satisfying

$$D(Ta, Tb) \leqslant s\big(D(a, Ta) + D(b, Tb)\big),$$

for all $a, b \in X$ such that $\frac{1}{K+1}D(a, Ta) \leqslant D(a, b)$.

Theorem 2.1.8. *Let (X, D, K) be a complete strong $b-$metric space and T be a Kannan-Suzuki mapping. Then T is a Picard operator.*

From the Theorem 2.1.8 we have the following corollary that a sufficient condition for mapping is a Picard operator on the complete metric spaces.

Corollary 2.1.9. *Let (X, ρ) be the complete metric space and $T : X \to X$ be a mapping. Suppose there exists $r \in [0, \frac{1}{2})$ satisfying*

$$\rho(Ta, Tb) \leqslant r\big(\rho(a, Ta) + \rho(b, Tb)\big), \tag{0.3}$$

with $a, b \in X$ such that $\frac{1}{2}\rho(a, Ta) \leqslant \rho(a, b)$. Then, T is a Picard operator.

It is obvious that, in Theorem 3, the necessary assumption (0.3) is satisfied for all $a, b \in X$, in our Corollary 2.1.9, condition (0.3) just satisfy with $a, b \in X$ such that $\frac{1}{2}\rho(a, Ta) \leqslant \rho(a, b)$. That is, all mappings satisfy Theorem 3 both satisfy the Corollary 2.1.9.

Definition 4. Let (X, D, K) be a strong $b-$metric space. We say that $T : X \to X$ is *Kannan-Suzuki type mapping* if

$$D(Ta, Tb) < \frac{1}{2}\big(D(a, Ta) + D(b, Tb)\big),$$

for all $a, b \in X$ such that $\frac{1}{K+1}D(a, Ta) < D(a, b)$.

Combining the contraction types of T. Suzuki and J. Górnicki we obtain the following result about the existence of Picard operator in compact strong $b-$metric spaces.

Theorem 2.1.13. *Let (X, D, K) be a compact strong $b-$metric space and T be a Kannan-Suzuki type mapping . Then, T has a unique fixed point $\bar{a} \in X$. Furthermore, if T is continuous, then T is Picard operator.*

The Example 2.1.16 shows that for T to be the Picard operator in the Theorem 2.1.13, the continuity of the mapping T cannot be dropped. In addition, it is easy to see that the mapping class satisfying J. Górnicki's theorem also satisfies the Kannan-Suzuki type mapping condition. Combined with Example 2.1.14 shows that Theorem 2.1.13 is an extension of the result of Górnicki.

Definition 5. Let (X, D, K) be a strong $b-$metric space and $k \in (0, \frac{1}{2})$. We say that mapping $T : X \to CB(X)$ is *multivalued Kannan-Suzuki mapping* if there exists $s \in (0, k)$ satisfying

$$H(Ta, Tb) \leqslant s\big(d(a, Ta) + d(b, Tb)\big), \tag{0.4}$$

for all $a, b \in X$ such that $\frac{1}{K+1}d(a, Ta) \leqslant D(a, b)$.

In 2021, we extend L. S. Dube and S. P. Singh's Theorem 7 in complete strong $b-$metric space under the condition of a multivalued Kannan-Suzuki mapping.

Theorem 2.2.2. *Let (X, D, K) be a complete strong $b-$metric space and let T be a Kannan-Suzuki mapping multivalue. Then, T is a multivalued weak Picard operator.*

It can be seen that, in Theorem 7, the necessary assumption (0.4) is satisfied for every $a, b \in X$, in our Theorem 2.2.2, condition (0.4) just satisfy with $a, b \in X$ such that $\frac{1}{K+1}d(a, Ta) \leqslant D(a, b)$. That is, all mappings satisfy Theorem 7 satisfies the Theorem 2.2.2. Combined with Example 2.2.3 shows that Theorem 2.2.2 is an extension of the result of L. S. Dube and S.P. Singh.

(3) *For a third purpose:* We introduce the concept of a strong b-TVS cone metric space and study some properties for this space. Furthermore, we extend the results of Sh. Rezapour and

R. Hamlbarani for the strong b-TVS cone metric space with the order induced by the cone ($\preceq$) and prove the completion theorem for this class of spaces. The specific results are as follows:

Theorem 3.3.1. *Let (X, E, C, K, ρ) be a complete strong b-TVS cone metric space and mapping $T : X \to X$. Suppose there exists $s \in [0, 1)$ satisfying*

$$\rho(Ta, Tb) \preceq s\rho(a, b) \text{ for all } a, b \in X.$$

Then, T is a Picard operator.

Theorem 3.3.3. *Let (X, E, C, K, ρ) be a complete strong b-TVS cone metric space and mapping $T : X \to X$. Suppose there exists $s \in [0, \frac{1}{2})$ satisfying*

$$\rho(Ta, Tb) \preceq s\big(\rho(a, Ta) + \rho(b, Tb)\big) \text{ for all } a, b \in X.$$

Then, T is a Picard operator.

Note that, Theorem 3.3.1 is an extension of Theorem 2.3 and Theorem 3.3.3 is an extension of Theorem 2.6 in the work of Sh.Rezapour and R. Hamlbarani in 2008. This is illustrated by Example 3.3.2 and Example 3.3.4.

Using the neighborhood property of cones in a real Hausdorff local convex topological vector space, we establish the completion principle for the strong b-TVS cone metric space.

Theorem 3.4.7. *Let (X, E, C, K, ρ) be a strong b-TVS cone metric space and cone C satisfy the neighborhood property in the complete real Hausdorff local convex topological vector space E. Then*

(i) *(X, E, C, K, ρ) has a completion;*

(ii) *The completion of (X, E, C, K, ρ) is unique in the sense that if $(X_1^*, E, C, K_1, \rho_1^*)$ and $(X_2^*, E, C, K_2, \rho_2^*)$ are two completions of (X, E, C, K, ρ) then there is a bijective isometry $\phi : X_1^* \to X_2^*$ which restricts to the identity on X.*

Our above results answer the question of W. Kirk and N. Shahzad for the case in the strong b-TVS cone metric space. In addition, Example 3.4.8 shows that Theorem 3.4.7 is a generalize of Theorem 8.

The main results of our dissertation have been published in articles [A1], [A2], [A3], [A4], and [A5] in the list of publications dissertation-related work.

4. Research Methods

We use basic research methods: On the basis of a research-oriented literature review, we discover open problems that need to be solved and use our knowledge, techniques of Functional Analysis, fixed point theory, and differential equation theory to solve the problems posed.

In addition to publication in journals, the main results of the thesis have been reported at:
- Seminar of Calculus Department, Faculty of Mathematics, University of Education, Thai Nguyen University.
- The 19th Scientific Computation and Optimization Seminar, Ba Vi, April 22-24, 2021.

Author

Chapter 1

The weakly Picard operators in the complete metric spaces

In this chapter, we establish some sufficient conditions for mapping is to single-valued and multivalued weakly Picard operators in complete metric spaces. The main results of this chapter are published by us in the article [A2] in the List of published scientific publications related to the thesis.

1.1 The single-valued weakly Picard operators

Theorem 1.1.1. *Let (X, ρ) be a complete metric space and $T : X \to X$ be a mapping. Suppose there exists $\alpha > 0$ such that*

$$\frac{1}{2}\rho(a, Ta) \leqslant \rho(a, b) \text{ implies } \rho(Ta, Tb) \leqslant M(a, b, \alpha)\rho(a, b),$$

for all $a, b \in X$, where $M(a, b, \alpha) = \dfrac{\rho(a, Tb) + \rho(b, Ta) + \rho(a, b)}{2\rho(a, Ta) + \rho(b, Tb) + \alpha}$. Then

(1) *T is a weakly Picard operator;*

(2) *If $\bar{a}, \bar{b} \in X$ are two distinct fixed points of T, then $\rho(\bar{a}, \bar{b}) \geqslant \dfrac{\alpha}{3}$.*

Example 1.1.2. Let X be a set has least two elements. The function $\rho(a, b)$ is defined by

$$\rho(a, b) = \begin{cases} 0 & \text{if } a = b, \\ \frac{1}{3} & \text{if } a \neq b. \end{cases}$$

Then (X, ρ) is a complete metric space.

The mapping $T : X \to X$ defined by $Ta = a$ for all $a \in X$. It is easy to check that T does not satisfy Theorem 6. However, mapping T satisfies the conditions of Theorem 1.1.1 with $\alpha = 1$. Furthermore, T is a weakly Picard operator and if $\bar{a}, \bar{b}$ are two distinct fixed points of T then $\rho(\bar{a}, \bar{b}) \geqslant \dfrac{1}{3}$.

Example 1.1.3. Suppose $X = \{0, 1, 2\}$. The metric $\rho : X \times X \to [0, +\infty)$ defined by

$$\rho(a, a) = 0 \text{ with } a \in X,\ \rho(0, 1) = \rho(1, 0) = \frac{1}{2}, \rho(0, 2) = \rho(2, 0) = \frac{3}{2}, \rho(1, 2) = \rho(2, 1) = 2.$$

Then (X, ρ) is a complete metric space. Consider the mapping $T : X \to X$ by $T0 = 0, T1 = 1$ and $T2 = 1$. With $\alpha = 1$, by direct calculation, we see that T satisfies the conditions of

Theorem 1.1.1. It is easy to see that T is a weakly Picard operator with a set of fixed points $\{0,1\}$ and $\frac{1}{2} = \rho(0,1) \geqslant \dfrac{\alpha}{3} = \dfrac{1}{3}$.

Corollary 1.1.4. *Let (X,ρ) be a complete metric space and T be the mapping from X to itself. Suppose there exists $\alpha > 0$ such that T satisfies the assumption of Theorem 1.1.1. Then T has a unique fixed point if $M(a,b,\alpha) < 1$ for all $a,b \in X$.*

Example 1.1.5. Suppose $X = \{0,1,2\}$ and $p : X \times X \to \mathbb{R}$ defined by

$$\rho(0,0) = \rho(1,1) = \rho(2,2) = 0, \rho(0,1) = \rho(1,0) = \frac{1}{2}, \rho(0,2) = \rho(2,0) = 1,$$

$\rho(1,2) = \rho(2,1) = \frac{1}{2}$. Then (X,ρ) is a complete metric space. Consider the mapping $T : X \to X$ by $T0 = 0, T1 = 0$ and $T2 = 0$. For $\alpha = 2$, by direct calculation we have $M(a,b,\alpha) < 1$ for all $a,b \in X$. Furthermore, since $\rho(Ta,Tb) = 0$ for all $a,b \in X$, so

$$\rho(Ta,Tb) \leq M(a,b,2).\rho(a,b), \text{ for all } a,b \in X.$$

So, T satisfies the conditions of Corollary 1.1.4 with $\alpha = 2$. It is easy to see that T has a unique fixed point $\bar{a} = 0$.

Theorem 1.1.6. *Let (X,ρ) be a complete metric space and $T : X \to X$ be a mapping. Suppose there exists $\alpha > 0$ satisfying*

$$\frac{1}{2}\rho(a,Ta) \leqslant \rho(a,b) \text{ implies } \rho(Ta,Tb) \leqslant N(a,b,\alpha)\rho(a,b) \text{ for all } a,b \in X,$$

where $N(a,b,\alpha) = \dfrac{\rho(a,Tb) + \rho(b,Ta) + \rho(a,Ta) + \rho(b,Tb) + \rho(a,b)}{3\rho(a,Ta) + 2\rho(b,Tb) + \alpha}$. *Then*

(1) *T is a weakly Picard operator;*

(2) *If $\bar{a}, \bar{b} \in X$ are two distinct fixed points of T, then $\rho(\bar{a},\bar{b}) \geqslant \dfrac{\alpha}{3}$.*

1.2 The Multivalued weak Picard operators

Definition 1.2.1. Let two sets A, B, and 2^B be the family of all subsets of B. A mapping T going from the set A to the set 2^B is called a multivalued mapping from A to B, denoted by $T : A \to 2^B$.

Definition 1.2.2. Let a multivalued mapping $T : X \to 2^X$. The point $a_0 \in X$ is called a fixed point of the mapping T if $a_0 \in Ta_0$.

Lemma 1.2.3. *Let (X,ρ) be a metric space and $A, B \in CB(X)$. Then if $H(A,B) > 0$ then for each $q > 1$ and $a \in A$, there exists $b \in B$ such that*

$$\rho(a,b) < q \cdot H(A,B).$$

Theorem 1.2.4. *Let (X,ρ) be a complete metric space and $T : X \to CB(X)$ be a multivalued mapping. Suppose there exists $\alpha > 0$ satisfying*

$$\frac{1}{2}d(a,Ta) \leqslant \rho(a,b) \text{ implies } H(Ta,Tb) \leqslant P(a,b,\alpha)\rho(a,b) \text{ for all } a,b \in X,$$

$P(a,b,\alpha) = \dfrac{d(a,Tb) + d(b,Ta) + \rho(a,b)}{2\delta(a,Ta) + \delta(b,Tb) + \alpha}$, $\delta(a,A) := \sup_{b \in A} \rho(a,b)$. *Then*

(1) *T is a multivalued weakly Picard operator;*

(2) *If $\bar{a}, \bar{b} \in X$ are two fixed points of T then $\rho^2(\bar{a},\bar{b}) \geqslant \dfrac{\alpha}{3}H(T\bar{a},T\bar{b})$.*

Example 1.2.5. Suppose $X = \{0, 1, 2\}$ and $\rho : X \times X \to [0, +\infty)$ are defined by

$$\rho(a, b) = \begin{cases} 0, \text{ if } a = b \in X, \\ 2, \text{ if } a \neq b \in X. \end{cases}$$

Then (X, ρ) is a complete metric space. The mapping $T : X \to CB(X)$ defined by $T0 = \{0\}, T1 = \{1\}$, $T2 = \{1, 2\}$. With $\alpha = 2$, it is easy to check that T satisfies all conditions of Theorem 1.2.4 with $\alpha = 2$. Obviously, T is a multivalued weak Picard operator with fixed points of $\{0, 1, 2\}$ and if $\bar{a}, \bar{b}$ are two different fixed points of T then $\rho^2(\bar{a}, \bar{b}) \geqslant \frac{2}{3} H(T\bar{a}, T\bar{b})$.

Corollary 1.2.6. *Let (X, ρ) be a complete metric space and $T : X \to CB(X)$ be a multivalued mapping. Suppose there exists $\alpha > 0$ such that T satisfies the assumption of Theorem 1.2.4. Then T has a unique fixed point if $P(a, b, \alpha) < 1$ for all $a, b \in X$.*

Theorem 1.2.7. *Let (X, ρ) be a complete metric space and $T : X \to CB(X)$ be a multivalued mapping. Suppose there exists $\alpha > 0$ satisfying*

$$\frac{1}{2} d(a, Ta) \leqslant \rho(a, b) \text{ implies } H(Ta, Tb) \leqslant Q(a, b, \alpha)\rho(a, b) \text{ for all } a, b \in X,$$

where $Q(a, b, \alpha) = \dfrac{d(a, Tb) + d(b, Ta) + d(a, Ta) + d(b, Tb) + \rho(a, b)}{3\delta(a, Ta) + 2\delta(b, Tb) + \alpha}$. *Then*

(1) *T is a multivalued weakly Picard operator;*

(2) *If $\bar{a}, \bar{b} \in X$ are two fixed points of T then $\rho^2(\bar{a}, \bar{b}) \geqslant \frac{\alpha}{3} H(T\bar{a}, T\bar{b})$.*

Chapter 2

Picard and weakly Picard operators in strong $b-$metric spaces

In this chapter, we establish some of the sufficient conditions for mapping are single-valued Picard operator and multivalued weak Picard operator in strong $b-$metric space. The main results of this chapter are published by us in the articles [A1] and [A4] in the List of published scientific publications related to the thesis.

2.1 Single-valued Picard operators

In this section, we extend the results of J. Gócrnicki and R. Kannan to strong $b-$metric spaces. First, we recall some concepts in the strong $b-$metric spaces.

Definition 2.1.1. Let X be a non-empty set and the real number $K \geqslant 1$. A mapping $D : X \times X \to [0; +\infty)$ is said to be strong $b-$metric on X if:

(D1) $D(a, b) = 0$ if and only if $a = b$;

(D2) $D(a, b) = D(b, a)$ for all $a, b \in X$;

(D3) $D(a, b) \leqslant D(a, c) + KD(c, b)$ for all $a, b, c \in X$.

Then the triple (X, D, K) is called a strong $b-$metric space.

Definition 2.1.2. Let (X, D, K) is a strong $b-$metric space, $\{a_n\}$ is the sequence of elements in X and $a \in X$. Then:

(i) The sequence $\{a_n\}$ is said to converge to a if $\lim\limits_{n\to\infty} D(a_n, a) = 0$. We denote $\lim\limits_{n\to\infty} a_n = a$ or $a_n \to a$ as $n \to \infty$.

(ii) $\{a_n\}$ is called a Cauchy sequence in X if $\lim\limits_{n,m\to\infty} D(a_n, a_m) = 0$.

(iii) The strong $b-$metric space (X, D, K) is called a complete strong $b-$metric space if every Cauchy sequence in X is convergent.

(iv) The strong $b-$metric space (X, D, K) is said to be a compact strong $b-$metric space if every sequence in X contains a convergent subsequence.

Proposition 2.1.3. *Let $\{a_n\}$ be a sequence of elements in the strong $b-$metric space (X, D, K) and suppose*

$$\sum_{i=1}^{\infty} D(a_i, a_{i+1}) < \infty.$$

Then $\{a_n\}$ is a Cauchy sequence.

2.1.1. Picard operators for some Kannan-type mapping classes for control functions

Theorem 2.1.4. *Let (X, D, K) be a complete strong $b-$metric space and mapping T from X to itself. Suppose that there exists a function $f \in \mathcal{S}$ such that for each $a, b \in X$ and $a \neq b$, we always have*

$$D(Ta, Tb) \leqslant f(D(a, b))\big(D(a, Ta) + D(b, Tb)\big).$$

Then, T is a Picard operator.

Example 2.1.5. Take $X = \{0, 1, 2\}$ and $D : X \times X \to [0, +\infty)$ defined by

$$D(0,0) = D(1,1) = D(2,2) = 0,$$

$$D(0,1) = D(1,0) = \frac{1}{2},$$
$$D(0,2) = D(2,0) = 6,$$
$$D(1,2) = D(2,1) = 5.$$

The mapping $T : X \to X$ defined by $T0 = 0, T1 = 0, T2 = 1$, the function $f \in \mathcal{S}$ defined by $f(t) = \frac{1}{2}e^{\frac{-t}{6}}, t > 0$ and $f(0) \in [0, \frac{1}{2})$. Then $(X, D, K = 2)$ is a complete strong $b-$metric space but not a metric space because

$$6 = D(2,0) > D(2,1) + D(1,0) = \frac{11}{2}.$$

Hence, Theorem 4 does not apply. On the other hand, it is easy to see that the mapping T satisfies all the conditions of the Theorem 2.1.4 and T is a Picard operator.

Theorem 2.1.6. *Let (X, D, K) be a complete strong $b-$metric space and mapping T from X to itself. Suppose that there exists a function $\varphi \in \mathcal{H}$ such that for each $a, b \in X$ and $a \neq b$, we always have*

$$D(Ta, Tb) \leqslant \varphi(D(a, b))\big(D(a, Ta) + D(b, Tb) + D(a, b)\big),$$

Then, T is a Picard operator.

Example 2.1.7. Consider the complete strong $b-$metric space (X, D, K) and mapping T in Example 2.1.5. Obviously, Theorem 5 does not apply.

We construct the function $\varphi \in \mathcal{H}$ by $\varphi(t) = \frac{1}{3}e^{\frac{-t}{6}}, t > 0$ and $\varphi(0) \in [0, \frac{1}{3})$. Thus, the mapping T satisfies all the conditions of the Theorem 2.1.6. Obviously, T is a Picard operator.

2.1.2. Picard operators for Kannan-Suzuki mapping

Theorem 2.1.8. *Let (X, D, K) be a complete strong $b-$metric space and T be a Kannan-Suzuki mapping. Then, T is a Picard operator.*

Corollary 2.1.9. *Let (X,ρ) be a complete metric space and let $T : X \to X$ be a mapping. Suppose there exists $s \in [0, \frac{1}{2})$ satisfying*

$$\rho(Ta, Tb) \leqslant s\big(\rho(a, Ta) + \rho(b, Tb)\big),$$

for all $a, b \in X$ such that $\frac{1}{2}\rho(a, Ta) \leqslant \rho(a, b)$. Then, T is the Picard operator.

Theorem 2.1.10. *Suppose $f : [0,1]\times X \to \mathbb{R}$ is a real continuous function satisfying the following conditions:*

$$|f(t,a) - f(t,b)| \leqslant k|a(t) - b(t)| \tag{2.1}$$

for all $(t,a), (t,b) \in [0,1] \times X$ and

$$|f(t,a)| \leqslant k \text{ for all } (t,a) \in [0,1] \times X. \tag{2.2}$$

Then, the problem of the Cauchy differential equation

$$\begin{cases} \frac{da}{dt} = f(t,a) \\ a(0) = a_0 \end{cases} \tag{2.3}$$

has a unique solution $\bar{a}$ on X.

The class of parameter-dependent control functions $\mathcal{F}_q$ defined by:

$$\mathcal{F}_q = \{\psi : (0,\infty) \to [0,q) : \psi(t_n) \to q \text{ pull } t_n \to 0 \text{ as } n \to \infty\},$$

where $q \in (0, \frac{1}{2})$. Using the parameter dependency control function for the Kannan-Suzuki mapping, we obtain the following result for a strong $b-$metric space.

Theorem 2.1.11. *Let (X, D, K) be a complete strong $b-$metric space and mapping $T : X \to X$. Suppose there exists a function $\psi \in \mathcal{F}_q$ satisfying*

$$\frac{1}{K+1} D(a, Ta) \leqslant D(a, b)$$

implies

$$D(Ta, Tb) \leqslant \psi(D(a,b))\big(D(a, Ta) + D(b, Tb)\big),$$

for all $a, b \in X, a \neq b$. Then, T is a Picard operator.

Example 2.1.12. Let $X = \{0,1,2\}$ and the function $D : X \times X \to [0,+\infty)$ defined by $D(a,b) = (a-b)^2$. Then $(X, D, K = 3)$ is a complete strong $b-$metric space.

The mapping $T : X \to X$ defined by $T0 = 1, T1 = 1, T2 = 0$ and the function ψ defined by $\psi(t) = \frac{1}{3}e^{\frac{-t}{8}}, t > 0, \psi(0) \in [0, \frac{1}{3})$. Then $\psi \in \mathcal{F}_{\frac{1}{3}}$. It is easy to check that T satisfies the conditions of Theorem 2.1.11. Obviously, T is a Picard operator.

2.1.3. Picard operators for Kannan-Suzuki type mapping

Theorem 2.1.13. *Let (X, D, K) be a compact strong $b-$metric space and let T be a Kannan-Suzuki type mapping. Then, T has a unique fixed point $\bar{a} \in X$. Furthermore, if T is continuous, then T is a Picard operator.*

Example 2.1.14. Let $X = \{0,1,2\}$ and the function $D : X \times X \to [0,+\infty)$ defined by

$$D(0,0) = D(1,1) = D(2,2) = 0,$$

$$D(0,1) = D(1,0) = \frac{1}{2}$$
$$D(0,2) = D(2,0) = 6,$$
$$D(1,2) = D(2,1) = 5.$$

The mapping $T : X \to X$ defined by $T0 = 0, T1 = 0$ and $T2 = 1$. It is easy to see that $(X, D, K = 2)$ is a compact strong $b-$metric space but not a compact metric, because

$$6 = D(2,0) > D(2,1) + D(1,0) = \frac{11}{2}.$$

Therefore, Theorem 2.2 of Górnicki does not apply. However, it is easy to check that T satisfies the conditions of the Theorem 2.1.13 and that T has a unique fixed point $\bar{a} = 0$. Furthermore, for each $a \in X$ then $T^n a = 0$ for every $n \geqslant 2$. Hence, T is a Picard operator.

Corollary 2.1.15. *Let (X, ρ) be a compact metric space and $T : X \to X$ be a mapping satisfying*

$$\rho(Ta, Tb) < \frac{1}{2}\big(\rho(a, Ta) + \rho(b, Tb)\big)$$

for all $a, b \in X$ such that $\frac{1}{2}\rho(a, Ta) < \rho(a, b)$. Then, T has a unique fixed point $\bar{a} \in X$. Furthermore, if T is continuous then T is a Picard operator.

Example 2.1.16. Let $X = [-4, -3] \cup \{0\} \cup [3, 4]$, the function $D : X \times X \to [0, +\infty)$ defined by $\rho(a, b) = |a - b|$ for all $a, b \in X$. It is easy to see that (X, ρ) is a compact metric space. The mapping $T : X \to X$ defined by

$$Ta = \begin{cases} \dfrac{4a+9}{a+2}, & , \quad \text{if } a \in [-4, -3), \\ 0, & \text{if } a \in \{-3, 0, 3\}, \\ \dfrac{-4a+9}{a-2}, & \text{if } a \in (3, 4], \end{cases}$$

By direct calculation, we see that T satisfies all the conditions in Corollary 2.1.15 except the continuity of the mapping T. It is easy to see that T has a unique fixed point $\bar{a} = 0$. Furthermore, if $a \notin \{-3, 0, 3\}$ then the sequence $\{T^n a\}$ does not converge to $\bar{a} = 0$. Hence, T is not a Picard operator.

The proof is similar to the proof of Theorem 2.1.13, we get the following result.

Theorem 2.1.17. *Let (X, D, K) be a compact strong $b-$metric space and the mapping $T : X \to X$ satisfying*

$$D(Ta, Tb) < \frac{1}{3}\big(D(a, Ta) + D(b, Tb) + D(a, b)\big)$$

for $a, b \in X$ such that $\frac{1}{K+1}D(a, Ta) < D(a, b)$. Then, T has a unique fixed point $\bar{a} \in X$. Furthermore, if T is a continuous mapping, then T is the Picard operator.

2.2 Weakly Picard operator for multivalued Kannan-Suzuki mapping

Let (X, D, K) be a strong $b-$metric space, denoted $CB(X)$ be the collection of all non-empty, closed, and bounded subsets of X. The function H defined by

$$H(A, B) := \max\{\sup_{a \in B} d(a, A); \sup_{a \in A} d(a, B)\},$$

where $A, B \in CB(X)$ and $d(a, A) := \inf_{b\in A} D(a, b)$ is called *metric Hausdorff* on $CB(X)$ induced by D.

The proof is similar to the proof of Lemma 1.2.3, we get.

Lemma 2.2.1. *Suppose that (X, D, K) is a strong $b-$metric space and $A, B \in CB(X)$. If $H(A, B) > 0$ then for each $h > 1$ and $a \in A$ there is a $b \in B$ such that*

$$D(a, b) < h \cdot H(A, B).$$

Theorem 2.2.2. *Let (X, D, K) be a complete strong $b-$metric space, $T : X \to CB(X)$ be a multivalued Kannan-Suzuki mapping. Then T is a multivalued weakly Picard operator.*

Example 2.2.3. Suppose $X = \{1, 2, 3\}, K = 3$ and $D : X \times X \to [0, \infty)$ defined by

$$D(1, 2) = 1, D(1, 3) = 4, D(2, 3) = 2 \text{ and } D(1, 1) = D(2, 2) = D(3, 3) = 0.$$

The mapping $T : X \to CB(X)$ defined by

$$T1 = \{2\}, T2 = \{2\}, T3 = \{1, 2\}.$$

Then (X, D, K) is a complete strong $b-$metric space but not a metric space because

$$D(1, 3) > D(1, 2) + D(2, 3).$$

Therefore, Theorem 7 does not apply. On the other hand, by direct calculation, we see that T satisfies all conditions of Theorem 2.2.2 and T is a multivalued weakly Picard operator.

Chapter 3

Picard operators and completion principle for strong b-TVS cone metric spaces

In this chapter, we introduce the strong b-TVS cone metric space. We then extend the result of Sh. Rezapour and R. Hamlbarani in this space. Furthermore, we proved that any strong b-TVS cone metric space has a completion. The main results of this chapter are published by us in the article [A3] and [A5] in the List of published scientific publications related to the dissertation.

3.1 Neighborhood property of cones

Definition 3.1.1. Let E be a real Hausdorff localy convex topological vector spaces with its zero vector θ and $C \subset E$. We say that C is a cone in E if:

(i) C is closed, nonempty, $C \neq \{\theta\}$;

(ii) $ax + by \in C$ for all $x, y \in C$, and non-negative real numbers a, b;

(iii) $C \cap (-C) = \{\theta\}$.

Definition 3.1.2. Let E be a real Hausdorff local convex topological vector space and a cone $C \subset E$ with non-empty interior. We can define a partial order $\preceq$ on E with respect to C as follows

$$x, y \in E : x \preceq y \text{ if } y - x \in C.$$

If $x \preceq y$ and $x \neq y$ then we write $x \prec y$. If $y - x \in \operatorname{int} C$ then we write $x \ll y$.

In the following sections, we always suppose E is a real Hausdorff locally convex topological vector space, C is a cone in E with $\operatorname{int} C \neq \emptyset$ and $\preceq$ is partial ordering with respect to C. From the definition of the cone and the partial order on E, we can easily prove the following properties.

Proposition 3.1.3. *Let C be a cone in E. Then*

(i) $C + C = C$;

(ii) $\operatorname{int} C + C = \operatorname{int} C$;

(iii) $\operatorname{int} C + \operatorname{int} C = \operatorname{int} C$;

(iv) *For all $a \in \operatorname{int} C$ and $\beta > 0$ then $\beta a \in \operatorname{int} C$.*

Proposition 3.1.4. *Let C be a cone in E. Then*

(i) *If* $a \preceq b$ *and* $c \preceq d$ *then* $a + c \preceq b + d$ *for all* $a, b, c, d \in E$;

(ii) *If* $a \preceq b$ *and* $c \ll d$ *then* $a + c \ll b + d$ *for all* $a, b, c, d \in E$;

(iii) *If* $a \ll b$ *and* $c \ll d$ *then* $a + c \ll b + d$ *for all* $a, b, c, d \in E$;

(iv) *If* $a \preceq \lambda a, a \in C$ *and* $0 \leqslant \lambda < 1$ *then* $a = \theta$.

Proposition 3.1.5. *Let* C *be a cone in* E. *Then*

(i) *If* $a \preceq b$ *and* $b \ll c$ *then* $a \ll c$ *for all* $a, b, c \in E$;

(ii) *If* $a \ll b$ *and* $b \preceq c$ *then* $a \ll c$ *for all* $a, b, c \in E$;

(iii) *If* $a \ll b$ *and* $b \ll c$ *then* $a \ll b$ *for all* $a, b, c \in E$.

Proposition 3.1.6. *Suppose* $e \in \operatorname{int} C, \theta \preceq a_n$ *and* $\lim_{n\to\infty} a_n = \theta$. *Then there exists* $N \in \mathbb{N}$ *such that* $a_n \ll e$ *for all* $n > N$.

Definition 3.1.7. Let C be a cone in E. We say that C has neighborhood property if for any neighborhood U of θ in E, there is neighborhood V of θ in E such that

$$(V + C) \cap (V - C) \subset U.$$

Definition 3.1.8. Let C be a cone in E. We say that a set $B \subseteq E$ generates the cone C and write $C = \operatorname{cone}(B)$ if

$$C = \{tb : b \in B, t \geqslant 0\}.$$

If in addition B does not contain zero and for each $c \in C\backslash\{\theta\}$ there are unique $b \in B, t > 0$ such that $c = tb$, then we say that B is a base of C.

Remark 3.1.9. If C has a closed convex bounded base then C has neighborhood property.

Example 3.1.10. Take $E = \mathbb{R}, C = \mathbb{R}_+$. Then, it is easy to see that C has the neighborhood property.

Proposition 3.1.11. *Assume that the* C *has the neighborhood property. Then for any neighborhood* U *of* θ *in* E, *there is* $e \in E, \theta \ll e$ *such that*

$$C \cap (e - C) \subset U.$$

Lemma 3.1.12. *Suppose that* C *has the neighborhood property in a real Hausdorff locally convex topological vector space* E. *Let* $\{u_n\}$, $\{v_n\}, \{a_n\}$ *be sequences in* E *such that* $a_n \preceq u_n \preceq v_n$ *for all* $n \geqslant 1$ *and* $\lim_{n\to\infty} v_n = \lim_{n\to\infty} a_n = \theta$. *Then* $\lim_{n\to\infty} u_n = \theta$.

Example 3.1.13. Let $E = C^1_{[0,1]}$ with norm

$$\|f\| = \|f\|_\infty + \|f'\|_\infty,$$

and consider the cone $C = \{f \in E : f(t) \geqslant 0, \forall t \in [0,1]\}$. Then, C has no neighborhood property. Indeed, consider $f_n(t) = \frac{t^n}{n}$ and $g_n(t) = \frac{1}{n}$ for all $t \in [0,1]$. Then $\theta \preceq f_n \preceq g_n$ for all n and $\lim_{n\to\infty} g_n = \theta$. On the other hand, we have

$$\|f_n\| = \max_{t\in[0,1]} \frac{t^n}{n} + \max_{t\in[0,1]} t^{n-1} = 1 + \frac{1}{n} > 1 \text{ for all } n \geqslant 1.$$

Hence f_n does not converge to θ. By Lemma 3.1.12, C has no neighborhood property.

3.2 Strong b-TVS cone metric spaces

Definition 3.2.1. Let X be a nonempty set and $K \geqslant 1$. The mapping $\rho : X \times X \to E$ is called a strong b-TVS cone metric on X if

(d1) $\theta \preceq \rho(a, b)$ for all $a, b \in X$
and $\rho(a, b) = \theta$ if and only if $a = b$;

(d2) $\rho(a, b) = \rho(b, a)$ for all $a, b \in X$;

(d3) $\rho(a, b) \preceq \rho(a, c) + K\rho(c, b)$ for all $a, b, c \in X$.
Then (X, E, C, K, ρ) is called a strong b-TVS cone metric space.

Definition 3.2.2. Let (X, E, C, K, ρ) be a strong b-TVS cone metric space and $\{a_n\}$ be a sequence of X. We say that

(i) a is the limit of the sequence $\{a_n\}$ if for every $e \in E$, $\theta \ll e$, there exists a natural number n_0 such that $\rho(a_n, a) \ll e$ for all $n \geqslant n_0$. We denote this $a_n \to a$ or $\lim\limits_{n\to\infty} a_n = a$.

(ii) $\{a_n\}$ is a Cauchy sequence if for every $e \in E$, $\theta \ll e$, there is n_0 such that $\rho(a_n, a_m) \ll e$ for all $n, m \geqslant n_0$.

(iii) If every Cauchy sequence is convergent in X then (X, E, C, K, ρ) is called a complete strong b-TVS cone metric space.

Lemma 3.2.3. *Let (X, E, C, K, ρ) be a strong b-TVS cone metric space and $\{a_n\}$ be a sequence of X. Then we have:*

(i) *If $\{a_n\}$ converges to $a \in X$ then $\{a_n\}$ is a Cauchy sequence.*

(ii) *If $\{a_n\}$ converges to $a \in X$ and $\{a_n\}$ converges to $b \in X$ then $a = b$.*

Lemma 3.2.4. *Let (X, E, C, K, ρ) be a strong b-TVS cone metric space, cone C has the neighborhood property and $\{a_n\}$, $\{b_n\}$ are two sequences in X. Then*

(i) *$\lim\limits_{n\to\infty} a_n = a \in X$ if and only if $\lim\limits_{n\to\infty} \rho(a_n, a) = \theta$.*

(ii) *$\{a_n\}$ is a Cauchy sequence if and only if $\lim\limits_{n,m\to\infty} \rho(a_n, a_m) = \theta$.*

(iii) *If $\lim\limits_{n\to\infty} a_n = a \in X$ and $\lim\limits_{n\to\infty} b_n = b \in X$ then $\lim\limits_{n\to\infty} \rho(a_n, b_n) = \rho(a, b)$.*

Remark 3.2.5. Note that in the work of L. G. Huang and X. Zhang, Lemma 3.2.4 was proved with the condition that C is a normal cone in Banach space. Whereas in our own case, we suppose that the cone C has neighborhood property instead. Further, Lemma 3.2.4 may fail to hold if C has no neighborhood property in E. The following example illustrates this.

Example 3.2.6. Suppose $X = \{0, \frac{1}{n} : n \geqslant 1\}$ and $\rho : X \times X \to E$ defined by

$$\rho(a, b) = \begin{cases} \theta, \text{ if } a = b, \\ |f_n - f_m|, \text{ if } a \neq b \in \{\frac{1}{n}, \frac{1}{m}\}, \\ f_n, \text{ if } a \neq b \in \{\frac{1}{n}, 0\}, \end{cases}$$

where $f_n(t) = \frac{t^n}{n}$ for all $t \in [0, 1]$ and $n \geqslant 1$.

It is easy to check that ρ is a strong b-TVS cone metric on X with $K = 1$ and $(X, E, C, K = 1, \rho)$ is a strong b- TVS cone metric space. Furthermore, we have

$$\rho(\frac{1}{n}, 0) = f_n \preceq \frac{I_E}{n} \text{ for all } n \geqslant 1,$$

where $I_E \in E$ by $I_E(t) = t$ with $t \in [0,1]$. Since $\lim_{n\to\infty} \frac{I_E}{n} = \theta$, by Proposition 3.1.6, for all $e \in E, \theta \ll e$, there exists a natural number n_0 such that

$$\frac{I_E}{n} \ll e \text{ for all } n \geqslant n_0.$$

So, $\rho(\frac{1}{n}, 0) \ll e$ for all $n \geqslant n_0$. Hence $\lim_{n\to\infty} \frac{1}{n} = 0$. On the other hand, we have

$$\|\rho(\frac{1}{n}, 0) - \theta\| = \|f_n - \theta\| = \max_{t\in[0,1]} \frac{t^n}{n} + \max_{t\in[0,1]} t^{n-1} = 1 + \frac{1}{n} > 1 \text{ for all } n \geqslant 1.$$

Hence $\rho(\frac{1}{n}, 0)$ does not converge to θ in E. So Lemma 3.2.4(i) does not holds.

3.3 Picard operators in strong b-TVS cone metric spaces

Theorem 3.3.1. *Let (X, E, C, K, ρ) be a complete strong b-TVS cone metric space and mapping $T : X \to X$. Suppose there exists $s \in [0, 1)$ satisfied*

$$\rho(Ta, Tb) \preceq s\rho(a, b) \textit{ for all } a, b \in X.$$

Then, T is a Picard operator.

Example 3.3.2. Take $X = \{0, 1, 2\}, E = \mathbb{R}^2$ and

$$C = \{(a, b) \in E : a \geqslant 0, b \geqslant 0\}.$$

The function $\rho : X \times X \to E$ defined by

$$\rho(a, a) = \theta \text{ with } a \in X, \quad \rho(a, b) = \rho(b, a) \quad \text{for all } a, b \in X$$

and

$$\rho(0, 1) = (4, 4), \quad \rho(0, 2) = \rho(1, 2) = (1, 1).$$

The mapping $T : X \to X$ defined by $T1 = T2 = T0 = 0$. Obviously $(X, E, C, K = 3, \rho)$ is a complete strong b-TVS cone metric space but not a cone metric space because

$$\rho(0, 2) + \rho(2, 1) = (2, 2) \preceq (4, 4) = \rho(0, 1).$$

Therefore, the results of Sh. Rezapour and R. Hamlbarani does not apply. However, it is easy to check that all the assumptions of the Theorem 3.3.1 are satisfied and T is the Picard operator.

Theorem 3.3.3. *Let (X, E, C, K, ρ) be a complete strong b-TVS cone metric space and mapping $T : X \to X$. Suppose there exists $s \in [0, \frac{1}{2})$ satisfied*

$$\rho(Ta, Tb) \preceq s\big(\rho(a, Ta) + \rho(b, Tb)\big) \textit{ for all } a, b \in X.$$

Then, T is a Picard operator.

Example 3.3.4. Take $X = \{0, 2, 3\}, E = \mathbb{R}^2$ and

$$C = \{(a, b) \in E : a \geqslant 0, b \geqslant 0\}.$$

The function $\rho : X \times X \to E$ defined by

$$\rho(a, a) = \theta = (0, 0) \text{ with } a \in X, \; \rho(3, 0) = \rho(0, 3) = (3, 3),$$

$$\rho(2, 0) = \rho(0, 2) = \rho(2, 3) = \rho(3, 2) = (1, 1).$$

The map $T : X \to X$ defined by $T0 = T2 = T3 = 0$. Obviously, $(X, E, C, K = 2, \rho)$ is a complete strong b-TVS cone metric space but not a cone metric space. Therefore, the results of Sh. Rezapour and R. Hamlbarani do not apply. However, it is easy to check that all the assumptions of the Theorem 3.3.3 are satisfied and that T is the Picard operator.

3.4 Completion of strong b-TVS cone metric spaces

Definition 3.4.1. Let (X, E, C, K, ρ) be a strong b-TVS cone metric space. For $a_0 \in X$ and $e \in E, \theta \ll e$, the subset $B(a_0, e) := \{a \in X : \rho(a_0, a) \ll e\}$ of X will be called an open ball centered at a_0 with radius e. We say that

(i) A set $A \subset X$ is an open set if for any $a \in A$, there is $e_a \in E, \theta \ll e_x$ such that $B(a, e_a) \subset A$. A set $B \subset X$ is closed if it's complement is open.

(ii) The intersection of all closed sets containing A is the closure of A, denoted by $\bar{A}$. The set A is said to be dense in X if $\bar{A} = X$.

Remark 3.4.2. If (X, E, C, K, ρ) is a strong b-TVS cone metric space. Then the following statements are true:

(i) The empty set $\emptyset$ and the space X are open and closed.

(ii) The open ball $B(a_0, e)$ is open, the closed ball $\bar{B}(a_0, e)$ is closed, where

$$\bar{B}(a_0, e) := \{a \in X : \rho(a_0, a) \preceq e\}, e \in E, \theta \ll e, a_0 \in X.$$

(iii) The union of open sets is open, and the intersection of an arbitrary number of closed sets is closed.

(iv) The intersection of a finite number of open sets is open, and the union of a finite number of closed sets is closed.

Proposition 3.4.3. *The sequence of elements $\{a_n\}$ of X is convergent to a in (X, E, C, K, ρ) if and only if every open set W contains a, exists a natural number n_0 such that $a_n \in W$ for all $n \geqslant n_0$.*

Proposition 3.4.4. *Let A be a subset of the strong b-TVS cone metric space (X, E, C, K, ρ). Then, $a \in \bar{A}$ if and only if there exists a sequence $\{a_n\} \subset A$ such that $\lim_{n\to\infty} a_n = a$.*

Proposition 3.4.5. *Let A be a subset of the strong b-TVS cone metric space (X, E, C, K, ρ). Then A is dense in X if and only if for each $a \in X$ and $e \in E, \theta \ll e$, there exists $a \in A$ such that $\rho(a, A) \ll e$.*

Definition 3.4.6. (i) Map $f : X \to Y$ from a strong b-TVS cone metric space (X, E, C, K, ρ) into a strong b- TVS cone metric space (Y, E, C, K', ρ') is called an isometric if

$$\rho'(f(a), f(b)) = \rho(a, b) \text{ for all } a, b \in X.$$

(ii) A complete strong b-TVS strong cone metric space (X^*, E, C, K^*, ρ^*) is said to be a completion of the strong b-TVS cone metric (X, E, C, K, ρ) if there exists an isometric $f : X \to X^*$ such that $\overline{f(X)} = X^*$.

Theorem 3.4.7. *Let (X, E, C, K, ρ) be a strong b-TVS cone metric space and the cone C has neighborhood property in a complete real Hausdorff locally convex topological vector space E. Then*

(i) *(X, E, C, K, ρ) has a completion;*

(ii) *The completion of (X, E, C, K, ρ) is unique in the sense that if $(X_1^*, E, C, K_1, \rho_1^*)$ and $(X_2^*, E, C, K_2, \rho_2^*)$ are two completions of (X, E, C, K, ρ) then there is a bijective isometry $\phi : X_1^* \to X_2^*$ which restricts to the identity on X.*

Example 3.4.8. Let $X = \mathbb{Q}, E = \mathbb{R}^2, C = \{(t,0) \in E : t \geqslant 0\}, K \geqslant 1$ and $\rho : X \times X \to E$ by $\rho(a,b) = (|a-b|, 0)$ for all $a, b \in X$. Because (X, E, C, K, ρ) is not a strong b-metric space. Therefore Theorem 8 does not apply. We can easily check that (X, E, C, K, ρ) is a strong b-TVS cone metric space, and all the assumptions in Theorem 3.4.7 are satisfied. Furthermore, we have $(X^* = \mathbb{R}, E = \mathbb{R}^2, C, K, \rho^*)$ which is a completion of (X, E, C, K, ρ) , where $\rho^*(a^*, b^*) = (|a^* - b^*|, 0)$ for all $a^*, b^* \in X^*$.

GENERAL CONCLUSION

The dissertation studies the existence of some classes of Picard and weakly Picard operators on metric space and generalized metric.

The main results of the dissertation:

1. Establish some sufficient conditions for a single-valued or multi-valued mapping to be the a weakly Picard operator in complete metric spaces.

2. Establish some sufficient conditions for a single-valued mapping to be a Picard operator and a multi-valued mapping to be a weakly Picard operator in strong $b-$metric space.

3. Introduce the strong b-TVS cone metric space and establish some sufficient conditions for a single-valued mapping to be a Picard operator in this space.

4. Establish the completion principle of the strong b-TVS cone metric space.

We propose some further research directions for the results of the dissertation as follows:

1. Study the Picard and the weakly Picard operators for incompletely generalized metric spaces.

2. Study some applications of the research results relating to the Picard and weakly Picard operators to solved about the existence of solutions of differential equations, systems of linear equations, and integral equations.

3. Study the non-cooperative equilibrium problem in games on generalized metric spaces.

LIST OF THE SCIENTIFIC PUBLICATIONS RELATED OF DISSERTATION

[A1] Hieu Doan (2021), "A new type of Kannan's fixed point theorem in strong b-metric spaces", *AIMS Mathematics* , 6 (7), 7895–7908. (SCIE)

[A2] Doan Trong Hieu and Bui The Hung (2022), "Some fixed point theorems for weakly Picard operators in complete metric spaces and applications", *Commun. Korean Math. Soc.* Vol. 37, No. 1, 75–89. (ESCI, Scopus)

[A3] Doan Trong Hieu, Bui The Hung, Muhammad Sirajo Abdullahi, Poom Kumam (2022), "On Answer to Kirk-Shahzad's Question for Strong b-TVS cone metric spaces", *Science and Technology Asia,* Vol. 27, No. 1, 20–30. (Scopus)

[A4] Ha Tran Phuong, Bui The Hung and Doan Trong Hieu (2023), "Fixed point theorems of Kannan type contractive mappings in strong b-metric spaces", submitted to *Miskolc Mathematical Notes.* (SCIE)

[A5] Bui The Hung and Doan Trong Hieu (2023), "Picard operators in strong b-TVS cone metric spaces", submitted to *East-West Journal of Mathematics.*