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SOME PROPERTIES OF SOLUTIONS OF A CLASS OF EQUATIONS CONTAINING STRONGLY DEGENERATE ELLIPTIC OPERATORS

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Introduction

1. History and significance of the problem

Nonlinear partial differential equations appear in many physical, chemical and biological processes, such as heat transfer processes, wave propagation in fluid mechanics, chemical reactions, population modeling in biology, etc. The study of these classes of equations has an important role in science and technology, so it has been attracting the attention of many scientists around the world. Until the 20s of the 20th century, solutions to equations, systems of partial differential equations were most commonly understood as classical solutions, i. e. solutions requiring differentiability in the ordinary sense up to the order of the equations, this makes it very difficult to prove the correctness of this problem, especially the smoothness of solution depending on the geometric structure of the considered domain. The introduction of the concept of a generalized solution was a methodologically central turning point in the study of equations, systems of partial differential equations, and their related variational problems. Some domestic authors have achieved profound results in the study of equations, systems of nonlinear degenerate elliptic equations, degenerate parabolic equations and degenerate hyperbolic equations. The obtained results are: existence and non-existence of solutions of boundary value problems for degenerate elliptic equations, existence of solutions, asymptotic behavior of solutions to degenerate parabolic equations, existence of solutions, asymptotic behavior of solutions to degenerate hyperbolic equations. As we all know, one of the most studied elliptic operators is the Laplace operator in space $\mathbb{R}^N$:

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + ... + \frac{\partial^2 u}{\partial x_N^2}.$$

Research on the existence or non-existence of non-trivial solutions of semi-linear equations containing the Laplace operator has been carried on by many mathematicians since the mid-twentieth century.

S. I. Pohozaev considered the boundary value problem:

$$\begin{cases} \Delta u + f(u) = 0 \quad \text{in} \quad \Omega, \\ \qquad\qquad u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

with Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$,

$$f(u) = \lambda u + |u|^{t-1} u.$$

We see that $u = 0$ is the trivial solution of the problem. S. I. Pohozaev gave an identity which is now named after him. Up to now, the research on existence of non-trivial solutions of boundary value problems containing elliptic operators is relatively complete. Similar problems arise for equations containing degenerate elliptic operators.

In 2012, P.T. Thuy and N. M. Tri considered the Dirichlet problem for the following second-order degenerate elliptic equation

$$\begin{cases} -P_{\alpha,\beta}u + f(u) = 0 \ \text{ in } \ \ \Omega, \\ \qquad\qquad\qquad u = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{2}$$

where

$$P_{\alpha,\beta}u = \Delta_x u + \Delta_y u + |x|^{2t_1}|y|^{2t_2}\Delta_z u, \qquad \text{with} \quad \alpha, \beta \geq 0, \tag{3}$$

Ω is a bounded domain in $\mathbb{R}^{N_1+N_2+N_3}$, $x \in \mathbb{R}^{N_1}$, $y \in \mathbb{R}^{N_2}$,$z \in \mathbb{R}^{N_3}$, with smooth boundary $\partial\Omega$, $f(u) = u|u|^{\gamma-1}$. The condition for non-existence of non-trivial solutions of Problem (2) is $\gamma > \dfrac{\widetilde{N}_{\alpha,\beta}+2}{\widetilde{N}_{\alpha,\beta}-2}$ and Ω is P_{t_1,t_2}- star, $\widetilde{N}_{\alpha,\beta} = N_1 + N_2 + (1+\alpha+\beta)N_3$.

Also in the same year, A. E. Kogoj and E. Lanconelli, studying the more general second order degenerate elliptic equations containing the operator Δ_γ, gave Pohozaev's identity and proved the existence of weak solutions and smoothness of weak solutions of the following problem by the method of variation

$$\begin{cases} \Delta_\gamma u - \eta u + f(X,u) = 0 \quad \text{in} \quad \Omega, \\ \qquad\qquad\qquad\qquad u = 0 \ \text{ on } \ \ \partial\Omega, \end{cases} \tag{4}$$

where Ω is a bounded open set in $\mathbb{R}^N$, $\eta \geq 0$ and $\Delta_\gamma u$ will be defined later in Chapter I, Section 1.1.1.

In 2016, C. T. Anh and B. K. My studied the existence of solutions of Problem (4) with $\eta = 0$ under the condition that $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following assumptions:

- $f(X,0) = 0$, $\lim\limits_{|\xi|\to+\infty} \frac{f(X,\xi)}{|\xi|^{2^*_\gamma-1}} = 0$, $2^*_\gamma = \frac{2\widetilde{N}}{\widetilde{N}-2}$, $(2^*_\gamma - 1 = \frac{\widetilde{N}+2}{\widetilde{N}-2})$;
- $\lim\limits_{|\xi|\to+\infty} \frac{F(X,\xi)}{|\xi|^2} = 0$ uniformly in $X \in \overline{\Omega}$, where $F(X,u) = \int\limits_0^u f(X,\xi)d\xi$;
- $\limsup\limits_{|\xi|\to 0} \frac{F(X,\xi)}{|\xi|^2} < \mu_1$ uniformly in $X \in \overline{\Omega}$, with μ_1 being the first eigenvalue of the operator -Δ_γ in the domain Ω with the homogeneous Dirichlet boundary condition;
- There exist $C_* \geq 0, \theta > 0$ satisfying:

  $$H(X,s) \leq \theta H(X,t) + C_*, \forall s,t \in \mathbb{R}, 0 < |s| < |t|, \forall X \in \Omega,$$

  where $H(X,u) = \frac{1}{2}uf(X,u) - F(X,u)$.

Then Problem (4) always has a weak nontrivial solution. Besides, since the 80s of the last century, many authors have studied the fourth order semi-linear elliptic equation containing the Laplace operator Δ

$$\begin{cases} \Delta^2 u = f(x,u) = 0, \ x \in \Omega \subset \mathbb{R}^N, \\ u = \partial_\nu u = 0, \ x \in \partial\Omega, \end{cases} \tag{5}$$

where $f(x,0) = 0$. Some sufficient conditions for the behavior of the function $f(X,u)$ with respect to the variable u have been given by the authors to guarantee either the non-existence of a non-trivial solution of Problem (5) or the existence of at least one nontrivial solution. However, to our knowledge, there are no studies for fourth order degenerate elliptic equations.

In 2005, Fall Djiby by using the solution tail estimation method, has proved the existence of a global attractor in the space $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ of the following problem

$$\begin{cases} u_{tt} + \beta u_t + u = \Delta u + f(X,u), X \in \mathbb{R}^N, t > 0, \\ u(X,0) = u_0(X), u_t(X,0) = u_1(X), \end{cases}$$

where β is a positive constant, $u_0(X) \in H^1(\mathbb{R}^N), u_1(X) \in L^2(\mathbb{R}^N)$. Function $f(X,\xi)$ is defined by:

- $\xi - f(X,\xi) = \xi + h_1(\xi) - h_2(X), h_2(X) \in L^2(\mathbb{R}^N)$,
- $h_1 \in C^1(\mathbb{R},\mathbb{R}), h_1(0) = 0, h_1(\xi)\xi \geq CF(\xi) \geq 0, \forall \xi \in \mathbb{R}$, where C is a positive constant, $F(\xi) = \int\limits_0^\xi h_1(\tau)d\tau$
- $0 \leq \limsup\limits_{|\xi|\to\infty} \dfrac{h_1(\xi)}{\xi} < \infty$.

In 2014, A. E. Kogoj and S. Sonner studied the following problem for a class of degenerate hyperbolic equations

$$\begin{cases} u_{tt}(X,t) + \lambda u_t = P_{\alpha,\beta} u(X,t) + f(u(X,t)), \quad X \in \Omega, t > 0, \\ u(X,t) = 0, \quad X \in \partial\Omega, t > 0, \\ u(X,0) = u_0(X), u_t(X,0) = u_1(X), \end{cases} \tag{6}$$

where Ω is a bounded region with smooth boundary in $\mathbb{R}^N$, $P_{\alpha,\beta}$ is the operator defined in (3), λ is a positive constant and $f(\xi)$ satisfies the conditions:

- $|f(\xi_1) - f(\xi_2)| \leq C|\xi_1 - \xi_2|(1 + |\xi_1|^\rho + |\xi_2|^\rho)$, (7)
  with $C > 0$, $0 \leq \rho < \frac{2}{\widetilde{N}_{\alpha,\beta}-2}$, $\widetilde{N}_{\alpha,\beta} = N_1 + N_2 + (\alpha+\beta+1)N_3 - 2$;
- $\limsup\limits_{|t|\to+\infty} \frac{f(t)}{t} < \mu_1$, (8)
  with μ_1 being the first eigenvalue of the operator -$P_{\alpha,\beta}$ in the domain Ω with a homogeneous Dirichlet boundary condition.

Then Problem (6) has a global solution and a global attractor. From the above said, we see that there are some results for the class of degenerate elliptic equations, hyperbolic degenerate equations. However the results obtained are still few and many open problems need to be studied. The open problems that we are interested in this thesis include:

- Study conditions for existence and non-existence of non-trivial solutions of Dirichlet's problem for fourth order degenerate elliptic equations containing the operator Δ_γ^2 in some cases of nonlinear functions.
- Study the existence and uniqueness of global integral solutions; the large time behavior of the solution (through the notion of global attractors) of Problem (6) containing the degenerate elliptic operator $P_{\alpha,\beta}$ in the whole space and the existence of the global attractor when the equation has an additional term $\gamma(X)u$ and the right hand side $f(X,u)$ depends on the variable X.

For the above reasons, we have chosen the research topic for our thesis as "**Some properties of solutions of a class of equations containing strongly degenerate elliptic operators**".

2. Research purposes

• *Content 1*: Study conditions for the existence and non-existence of solutions of Dirichlet's problem for fourth order semi-linear equations containing the Δ_γ^2-Laplace operator in bounded domains with the following problems: non-existence of strong non-trivial solutions; existence of weak non-trivial solutions.

• *Content 2*: Study the large time asymptotic behavior of semi-linear damped hyperbolic equations containing strongly degenerate elliptic operators $P_{\alpha,\beta}$ in the whole space with the following problems: existence and uniqueness of global integral solutions; existence of a compact global attractor and its structure.

3. Object and scope of the study

The object of the thesis is to consider boundary value problems and the initial boundary value problems for differential equations containing the degenerate elliptic operator Δ_γ, which is defined by

$$\Delta_\gamma u = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(\gamma_j^2 \frac{\partial u}{\partial x_j}\right),$$

where γ_j are functions that satisfy some conditions that are described in details in the dissertation.

4. Research Methods

• To study the existence of weak solutions, we use the method of variation and general theorems of critical point theory.

• To study the non-existence of strong solutions, we establish suitable Pohozaev-type identities for the operator Δ_γ^2 and exploit the geometric structure of the domain under consideration.

• To study the unique existence of global integral solutions, we use methods and tools of Nonlinear Functional Analysis: Galerkin approximation method, suitable forms of compact lemmas, lemmas dealing with nonlinear terms. To study the asymptotic behavior of the solution and the existence of a global attractor, we use the methods of infinite dimensional dynamic system theory. In particular, these are the method of asymptotic a priori evaluation and the method of evaluating the tail part of the solutions.

5. The results obtained and the significance of the project

The thesis has achieved the following main results.

• For the Dirichlet problem in a bounded domain for a degenerate elliptic equation of the fourth order, sufficient conditions are given so that no strong non-trivial solutions exist; proofs are presented for the existence of a weak solution of the problem with a number of conditions on the growth of the nonlinear term. This is the content of Chapter 2.

• For a damped hyperbolic equation containing the strongly degenerate elliptic operator $P_{\alpha,\beta}$ in $\mathbb{R}^N$, sufficient conditions are presented for the existence and uniqueness of a global integral solution. Proofs are given for the existence of a compact global attractor and the descriptions of its structure. This is the content of Chapter 3.

The results of the thesis are new, have scientific significance and contribute to completing the study of the existence of solutions in bounded domains of the boundary value problems for degenerate elliptic equations of the fourth order and the asymptotic shape of the solution of damped hyperbolic equations containing degenerate elliptic operators in the whole space.

6. Thesis structure

In addition to the introduction, overview, conclusion, recommendation, list of published works and list of references section, the thesis consists of 3 chapters

Chapter 1: *Prepare some necessary knowledge for the following chapters.* In this chapter we present the operator Δ_γ, some properties, examples and some additional knowledge (some function spaces and some embedding theorems); present some results on the critical points; global attractors and their properties.

- Chapter 2: *Present existence and non-existence of non-trivial solutions of semi-linear Δ_γ^2-Laplace equations in bounded domains.* In this chapter, we study the existence and non-existence of non-trivial solutions to the Dirichlet problem for fourth order semi-linear equations containing the operator Δ_γ^2:

$$\Delta_\gamma^2 u = f(x,u) \quad \text{in} \quad \Omega, \quad u = \partial_\gamma u = 0 \quad \text{on} \quad \partial\Omega,$$

where ν is the unit outward normal at boundary points of Ω. This chapter consists of three parts: the first part presents the Pohozaev identity for the operator Δ_γ^2. The second part gives some results on the non-existence of non-trivial strong solutions. The third part deals with the existence of non-trivial weak solutions.

Chapter 3: *Show the large time behavior of solutions of damped semi-linear hyperbolic equations which contain strongly degenerate operators in the whole space.* In this chapter, we study the existence of a compact global attractor of the semi-group of Cauchy problems for semi-linear degenerate hyperbolic equations. This chapter consists of two parts. The first part presents the existence and uniqueness of global integral solutions. The second part gives the conditions for the right hand side and the linear term of the equation, then reduces the equation to a system of first order equations, and finally proves the existence of a global attractor in the space $S_1^2(\mathbb{R}^N)$.

Chapter 1

Preparatory knowledge

In this chapter, we present some concepts and results for the following chapters. Specifically we present: the definition of the degenerate elliptic operator Δ_γ, some function spaces, embedding results, some results of the method of variation and the theory of critical points, the mountains pass theorem, semi-groups, global attractors and some other complementary knowledge.

Chapter 2

Solution of the boundary value problem for a degenerate elliptic equation of the fourth order

In this chapter we replace the variable X in Chapter 1 with the variable x and study the Dirichlet problem for a fourth order equation containing the operator Δ_γ^2 as follows:

$$\begin{cases} \Delta_\gamma^2 u = f(x,u) & \text{in} \quad \Omega, \\ u = \partial_\nu u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\nu = (\nu_1, \cdots, \nu_N)$ is the unit outward normal on the boundary $\partial\Omega$ and $\Delta_\gamma^2 := \Delta_\gamma(\Delta_\gamma)$.

The obtained results are theorems on the non-existence of non-trivial strong solutions and the theorems on the existence of non-trivial weak solutions with some conditions imposed on the nonlinear function $f(x,\xi)$.

The content of this chapter is based on Article [2] in the list of the author's scientific works related to the thesis.

2.1 Pohozaev's identity and the non-trivial strong solution theorem

In this section we will additionally assume that $\gamma_i \in C^2(\mathbb{R}^N),\ j = 2, ..., N$ with $\widetilde{N} > 4$. Put

$$T := \sum_{i=1}^{N} \varepsilon_i x_i \partial_{x_i}. \tag{2.2}$$

The operator T can also be thought of as the vector $T = (\epsilon_1 x_1 \partial_{x_1}, ..., \epsilon_N x_N \partial_{x_N})$. The first major result in this section is the following proposition.

Proposition 2.1. *Suppose that* $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ *and* $\Delta_\gamma^2 u \in L^1(\Omega)$. *Then*

$$\int_\Omega \Big[T(u)\Delta_\gamma^2 u + T(\Delta_\gamma u)\Delta_\gamma u\Big] \mathrm{d}x = (\widetilde{N} - 2) \int_\Omega \langle \nabla_\gamma u, \nabla_\gamma \Delta_\gamma u\rangle \mathrm{d}x$$
$$+ \int_{\partial\Omega} \Big[T(u)\langle \nabla_\gamma \Delta_\gamma u, \nu_\gamma\rangle + T(\Delta_\gamma u)\langle \nabla_\gamma u, \nu_\gamma\rangle\Big] \mathrm{d}S$$
$$- \int_{\partial\Omega} \langle \nabla_\gamma u, \nabla_\gamma \Delta_\gamma u\rangle \langle T, \nu\rangle \mathrm{d}S, \tag{2.3}$$

where T *is the vector field,* $\langle \cdot, \cdot\rangle$ *stands for the Euclidean inner product,* $\nu = (\nu_1, \cdots, \nu_N)$ *is the unit outward normal on* $\partial\Omega$ *and* $\nu_\gamma = (\gamma_1\nu_1, \cdots, \gamma_N\nu_N)$, dS *is the area element of* $\partial\Omega$.

Definition 2.1. A domain Ω is called *δ_t-starshaped* with respect to the origin if $0 \in \Omega$ and $\langle T, \nu\rangle \geq 0$ at every point of $\partial\Omega$.

Definition 2.2. A function $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ is called *a strong solution* of Problem (2.1) if $\Delta_\gamma^2 u = f(x, u)$ in Ω, $u = \partial_\nu u = 0$ on $\partial\Omega$ and $f(u, u(x)) \in L^1(\Omega)$.

If $u \equiv 0$ then u is called *the trivial solution* of Problem (2.1).

Proposition 2.2. *Suppose that* $f(x, \xi) \equiv f(\xi)$ *and* $f(0) = 0$. *Suppose that* $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ *is a strong solution of Problem* (2.1). *Then the function* u *satisfies the identity*

$$\int_\Omega \Big(\widetilde{N}F(u) - \frac{\widetilde{N} - 4}{2} u f(u)\Big) \mathrm{d}x = \frac{1}{2} \int_{\partial\Omega} |\Delta_\gamma u|^2 \langle T, \nu\rangle \mathrm{d}S. \tag{2.4}$$

Theorem 2.1. *Suppose that* $f(x, \xi) \equiv f(\xi), f(0) = 0$ *and* Ω *is* δ_t*-starshaped with respect to the origin and*

$$\widetilde{N}F(\xi) - \frac{\widetilde{N} - 4}{2} \xi f(\xi) < 0, \quad \forall \xi \neq 0.$$

Then Problem (2.1) *has no non-trivial solution* $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$.

Theorem 2.2. *Suppose that* $f(x, \xi) \equiv |\xi|^{p-1}\xi$ *and* $\widetilde{N} > 4$, Ω *is* δ_t*-starshaped with respect to the origin and*

$$p > \frac{\widetilde{N} + 4}{\widetilde{N} - 4}.$$

Then Problem (2.1) *has no non-trivial solution* $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$.

Example 2.1. Suppose that $B_1(0) \subset \mathbb{R}^3$ is the ball of radius 1 with the center at the origin. Then the problem

$$\begin{cases} \left(\frac{\partial^2}{\partial x^2} + x^4 \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right)^2 u = |u|^3 u & \text{in} \quad B_1(0), \\ u = \frac{\partial}{\partial \nu} = 0 \quad \text{on} \quad \partial B_1(0) \end{cases} \tag{2.5}$$

has no strong non-trivial solution because here we have $\gamma_1 = 1,\ \gamma_2 = \gamma_3 = x^2,\ \widetilde{N} = 1+2(1+2) = 7,\ \frac{\widetilde{N}+4}{\widetilde{N}-4} = \frac{11}{3}$, $B_1(0)$ is δ_t-star shape with respect to the origin and the growth order of the right hand side is 4 which is greater than $\frac{11}{3}$.

2.2 Some results on existence of weak solutions of degenerate elliptic equations of the fourth order

In this section, we will further assume that $\gamma_j(x) \in C^1(\mathbb{R}^N),\ \widetilde{N} > 4$ and present some existence results of solutions and multiple solutions of Problem (2.1) with appropriate conditions of the nonlinear function $f(x,\xi)$.

Definittion 2.3. A function $u \in S^{2,2}_{\gamma,0}(\Omega)$ is called *a weak solution* of Problem (2.1) if the identity

$$\int\limits_\Omega \Delta_\gamma u \Delta_\gamma \varphi \mathrm{d}x - \int\limits_\Omega f\left(x, u(x)\right) \varphi \mathrm{d}x = 0$$

is satisfied for every $\varphi \in S^{2,2}_{\gamma,0}(\Omega)$. To find the weak solutions of Problem (2.1), we find critical points of the energy functional Φ defined as follows:

$$\Phi(u) = \frac{1}{2}\int\limits_\Omega |\Delta_\gamma u|^2 \mathrm{d}x - \int\limits_\Omega F(x,u)\mathrm{d}x. \tag{2.6}$$

Lemma 2.1. *Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that there exist $p \in (2, 2^\gamma_*),\ f_1(x) \in L^{p_1}(\Omega), f_2(x) \in L^{p_2}(\Omega),$ where $p_1/(p_1-1) < 2^\gamma_*, pp_2/(p_2-1) \le 2^\gamma_*, p_1 > \max\{1, \frac{2^\gamma_* p_2}{p_2(p-1)+2^\gamma_*}\}, p_2 > 1$ satisfying*

$$|f(x,\xi)| \le f_1(x) + f_2(x)\,|\xi|^{p-1} \quad \textit{almost everywhere in } \ \Omega \times \mathbb{R}$$

Then $\Phi_1(u) \in C^1(S^{2,2}_{\gamma,0}(\Omega), \mathbb{R})$ and

$$\Phi_1'(u)(v) = \int\limits_\Omega f(x,u)v\mathrm{d}x$$

for all $v \in S^{2,2}_{\gamma,0}(\Omega)$, where

$$\Phi_1\left(u\right) = \int\limits_\Omega F\left(x,u\right)\mathrm{d}x, F(x,\xi) = \int\limits_0^\xi f(x,\tau)\mathrm{d}\tau.$$

We assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying:

(A1) There exist $p \in (2, 2^\gamma_*),\ f_1(x) \in L^{p_1}(\Omega),\ f_2(x) \in L^{p_2}(\Omega)$, where $p_1/(p_1-1) < 2^\gamma_*,\ pp_2/(p_2 -$

$1) < 2^{\gamma}_{*}$, $p_1 > \max\{1, \frac{2^{\gamma}_{*} p_2}{p_2(p-1)+2^{\gamma}_{*}}\}$, $p_2 > 1$, such that

$$|f(x,\xi)| \le f_1(x) + f_2(x)\,|\xi|^{p-1} \quad \text{almost everywhere in} \quad \Omega\times\mathbb{R};$$

(A2) $\lim\limits_{\xi\to 0} \frac{f(x,\xi)}{\xi} = 0$, uniformly for $x \in \Omega$.

(A3) $\lim\limits_{|\xi|\to\infty} \frac{|F(x,\xi)|}{\xi^2} = \infty$, for almost every $x \in \Omega$,

$$F(x,\xi) \ge 0 \quad \text{for all } (x,\xi) \in \Omega\times\mathbb{R};$$

(A4) There are constants $\mu > 2$ and $r_1 > 0$ such that

$$\mu F(x,\xi) \le \xi f(x,\xi) \quad \text{for all } (x,\xi) \in \Omega\times\mathbb{R}, |\xi| \ge r_1;$$

(A'4) There are constants C_0, $r_2 > 0$ and $\kappa > \max\{1, \frac{\widetilde{N}}{2}\}$ such that

$$|F(x,\xi)|^{\kappa} \le C_0|\xi|^{2\kappa}\mathcal{F}(x,\xi), \quad \forall (x,\xi) \in \Omega\times\mathbb{R},\ |\xi| \ge r_2,$$

where

$$\mathcal{F}(x,\xi) = \frac{1}{2} f(x,\xi)\xi - F(x,\xi);$$

(A5) $f(x,\xi)$ is an odd function in ξ.

From Lemma 2.1 and the fact that f satisfies the condition (A1), then the energy functional Φ is well defined on $S^{2,2}_{\gamma,0}(\Omega)$. Furthermore, $\Phi \in C^1(S^{2,2}_{\gamma,0}(\Omega),\mathbb{R})$ and

$$\Phi'(u)(v) = \int\limits_{\Omega} \Delta_\gamma u \Delta_\gamma v \mathrm{d}x - \int\limits_{\Omega} f(x,u)\, v \mathrm{d}x$$

for all $v \in S^{2,2}_{\gamma,0}(\Omega)$. From the definitions of weak solutions and the derivative of Φ, the critical points of Φ are weak solutions of Problem (2.1). The main results of this section are the following two theorems.

Theorem 2.3. *Assume that f satisfies* (A1)-(A3) and (A4). *Problem* (2.1) *has a non-trivial weak solution. Furthermore, if the condition* (A5) *is satisfied, Problem* (2.1) *has infinitely many non-trivial weak solutions.*

Theorem 2.4. *Assume that f satisfies* (A1)-(A3) and (A'4). *Problem* (2.1) *has a non-trivial weak solution. Furthermore, if the condition* (A5) *is satisfied, Problem* (2.1) *has infinitely many non-trivial weak solutions.*

Lemma 2.2. *Assume that f satisfies* (A1), (A3) *and* (A4). *Then the energy functional Φ satisfies the condition $(C)_c$ for every $c \in \mathbb{R}$ over $S^{2,2}_{\gamma,0}(\Omega)$.*

Lemma 2.3. *Suppose that f satisfies the conditions* (A1), (A3) *and* (A'4). *Then Φ satisfies the condition $(C)_c$ for every $c \in \mathbb{R}$ over $S^{2,2}_{\gamma,0}(\Omega)$.*

Lemma 2.4. *Suppose that* (A1) *and* (A2) *are satisfied. Then there exist $\alpha, \rho > 0$ satisfying*

$$\Phi(u) \geq \alpha, \quad \forall u \in S^{2,2}_{\gamma,0}(\Omega), \ \ \|u\|_{S^{2,2}_{\gamma,0}(\Omega)} = \rho.$$

Lemma 2.5. *Assume that* (A1) *and* (A3) *are satisfied. Then, for each finite dimensional subspace $\widehat{\mathbb{X}} \subset S^{2,2}_{\gamma,0}(\Omega)$, there exists a number $R = R(\widehat{\mathbb{X}}) > 0$ such that*

$$\Phi(u) \leq 0, \quad \forall u \in \widehat{\mathbb{X}}, \ \ \|u\|_{S^{2,2}_{\gamma,0}(\Omega)} \geq R.$$

Example 2.2. The problem

$$\begin{cases} \left(\frac{\partial^2}{\partial x^2} + x^4\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right)^2 u = |u|^2 u \ \text{ in a bounded domain } \ \Omega \subset \mathbb{R}^N, \\ u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{2.7}$$

has infinitely many non-trivial weak solutions. Just like in Example 2.1 here we have $\gamma_1 = 1, \ \gamma_2 = \gamma_3 = x^2, \ \widetilde{N} = 7, \ \frac{\widetilde{N}+4}{\widetilde{N}-4} = \frac{11}{3}$ and the growth order of the right hand side is 3 which is less than $\frac{11}{3}$.

Chapter 3

Long time behavior of solutions to semilinear hyperbolic equations involving strongly degenerate elliptic differential operators

The content of this chapter is based on Article [1] in the list of the author's scientific works related to the thesis.

3.1 The unique existence of integral solutions

3.1.1. The problem and function spaces

In this chapter we study the following problem:

$$u_{tt} + \lambda u_t + \gamma(X)u = P_{\alpha,\beta}u + f(X,u), t > 0, \tag{3.1}$$

$$X = (x,y,z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3} := \mathbb{R}^N,$$

$$u(X,0) = u_0(X), u_t(X,0) = u_1(X), \tag{3.2}$$

where λ is a positive constant, $u_0(X) \in \mathbb{S}_1^2(\mathbb{R}^N),\ u_1(X) \in L^2(\mathbb{R}^N)$ and

$$P_{\alpha,\beta}u = \Delta_x u + \Delta_y u + |x|^\alpha |y|^\beta \Delta_z u,$$

$$\Delta_x := \sum_{i=1}^{N_1} \frac{\partial^2}{\partial x_i^2},\ \Delta_y := \sum_{j=1}^{N_2} \frac{\partial^2}{\partial y_j^2},\ \Delta_z := \sum_{l=1}^{N_3} \frac{\partial^2}{\partial z_l^2},\ u_t := \frac{\partial u}{\partial t},\ u_{tt} := \frac{\partial^2 u}{\partial t^2},$$

$$|x|^{2\alpha} := \left(\sum_{i=1}^{N_1} x_i^2\right)^\alpha,\ |y|^{2\beta} := \left(\sum_{j=1}^{N_2} y_j^2\right)^\beta,\ \alpha \geq 0,\ \beta \geq 0$$

$$\widetilde{N}_{\alpha,\beta} := N_1 + N_2 + (1+\alpha+\beta)N_3 > 2.$$

Suppose that $\gamma(X), f(X,\xi)$ satisfy the following conditions

i) $\gamma:\mathbb{R}^N \longrightarrow \mathbb{R}$ is a function that satisfies the following property:

(i_1) For each $\theta\in(0,\infty)$ there is a constant $C_\theta\in(0,\infty)$ such that for all $u\in\mathbb{S}_1^2(\mathbb{R}^N)$,

$$\int\limits_{\mathbb{R}^N} |\gamma(X)||u(X)|^2\mathrm{d}X \le \theta\,\|u\|^2_{\mathbb{S}_1^2(\mathbb{R}^N)} + C_\theta\,\|u\|^2_{L^2(\mathbb{R}^N)}\,. \tag{3.3}$$

(i_2) There is a constant $\lambda_0>0$ such that for all $u\in\mathbb{S}_1^2(\mathbb{R}^N)$,

$$\|\nabla_{\alpha,\beta}u\|^2_{L^2(\mathbb{R}^N)} + \int\limits_{\mathbb{R}^N}\gamma(X)|u(X)|^2\mathrm{d}X \ge \lambda_0\,\|u\|^2_{L^2(\mathbb{R}^N)}\,. \tag{3.4}$$

ii)

$$f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$$

$$(X,\xi)\mapsto\ f(X,\xi)$$

is a function satisfying the Carathéodory condition, i. e. for each $\xi\in\mathbb{R}$ the mapping $X\mapsto f(X,\xi)$ is Lebesgue measurable almost everywhere in $\mathbb{R}^N$, and the mapping $\xi\mapsto f(X,\xi)$ is continuous.

The primitive of f with respect to the variable u is defined by

$$F(X,\xi)=\int\limits_0^\xi f(X,\tau)\mathrm{d}\tau,$$

and f satisfies the following properties:

(f_1) $f(X,0)=h(X)\in L^2(\mathbb{R}^N)$;

(f_2) For all $X\in\mathbb{R}^N$ and $\xi_1,\xi_2\in\mathbb{R}$

$$|f(X,\xi_1)-f(X,\xi_2)|\le C_1|\xi_1-\xi_2|\left(g(X)+|\xi_1|^\rho+|\xi_2|^\rho\right) \tag{3.5}$$

with $0<\rho\le\frac{2}{\tilde{N}_{\alpha,\beta}-2}$ and $g:\mathbb{R}^N\to\mathbb{R}$ is a measurable function such that for all $u\in\mathbb{S}_1^2(\mathbb{R}^N)$

$$\int\limits_{\mathbb{R}^N}|g(X)|^2|u(X)|^2\mathrm{d}X\le C_2\,\|u\|^2_{\mathbb{S}_1^2(\mathbb{R}^N)}\,, \tag{3.6}$$

where C_1, C_2 are positive constants.

(f_3) There are measurable functions $g_1,g_2:\mathbb{R}^N\to\mathbb{R}$, $g_1,g_2\in L^1(\mathbb{R}^N)$ satisfying

$$f(X,u)u\le g_1(X) \text{ for a. e., } X\in\mathbb{R}^N, u\in\mathbb{R}, \tag{3.7}$$

$$F(X,u)\le g_2(X) \text{ for a. e., } X\in\mathbb{R}^N, u\in\mathbb{R}. \tag{3.8}$$

The notation $L^p_\nu(\mathbb{R}^N)$ stands for the set of all measurable functions $u:\mathbb{R}^N\to\mathbb{R}$ satisfying

$$\|u\|_{L^p_\nu(\mathbb{R}^N)} := \sup_{Y\in\mathbb{R}^N}\left(\int\limits_{B(Y)}|u(X)|^p\mathrm{d}X\right)^{\frac{1}{p}}<\infty,$$

where $Y \in \mathbb{R}^N$ and $B(Y) = \{X \in \mathbb{R}^N : Y < X < Y + 1_e, 1_e = (1,1,\ldots,1)\}$, $p \geq 1$. The following lemma shows a sufficient condition for the function γ so that it satisfies the condition (i_1).

Lemma 3.1. *Assume that $p > 1$ and $\phi \in L^p_\nu(\mathbb{R}^N)$.*

(i) If $p \geq \frac{\widetilde{N}_{\alpha,\beta}}{2}$ then there is a constant $C \in (0,\infty)$ satisfying

$$\int_{\mathbb{R}^N} |\phi(X)||u(X)|^2 \mathrm{d}X \leq C \|u\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)} \quad \text{for all} \quad u \in \mathbb{S}^2_1(\mathbb{R}^N).$$

(ii) If $p > \frac{\widetilde{N}_{\alpha,\beta}}{2}$ then for each $\theta \in (0,\infty)$ there is a constant $C_\theta \in (0,\infty)$ satisfying

$$\int_{\mathbb{R}^N} |\phi(X)||u(X)|^2 \mathrm{d}X \leq \theta \|u\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)} + C_\theta \|u\|^2_{L^2(\mathbb{R}^N)} \quad \text{for all} \quad u \in \mathbb{S}^2_1(\mathbb{R}^N).$$

The norm equivalence is shown in the following lemma.

Lemma 3.2. *Suppose that γ satisfies the condition $i)$ and $0 < \kappa < \lambda_0, 0 < \theta < 1$. Then for every $u \in \mathbb{S}^2_1(\mathbb{R}^N)$*

$$\widehat{C_1} \|u\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)} \leq \|\nabla_{\alpha,\beta} u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \gamma(X)|u(X)|^2 \mathrm{d}X - \kappa \|u\|^2_{L^2(\mathbb{R}^N)} \leq$$
$$\leq \widehat{C_2} \|u\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)},$$

where $\widehat{C_1} = \min\{(\lambda_0 - \kappa)(1-\theta)/2(\lambda_0 + \theta + C_\theta), (\lambda_0 - \kappa)/2\}, \widehat{C_2} = \max\{1+\theta, \theta + C_\theta\}$ and θ, C_θ defined in (3.3).

Lemma 3.3. *Suppose that γ satisfies the condition $i)$. For all $u, v \in \mathbb{S}^2_1(\mathbb{R}^N)$ define*

$$((u,v))_{\mathbb{S}^2_1(\mathbb{R}^N)} = (\nabla_{\alpha,\beta} u, \nabla_{\alpha,\beta} v)_{L^2(\mathbb{R}^N)} + (\gamma u, v)_{L^2(\mathbb{R}^N)}.$$

Then $((\cdot,\cdot))_{\mathbb{S}^2_1(\mathbb{R}^N)}$ is a scalar product on $\mathbb{S}^2_1(\mathbb{R}^N)$ and the norm defined by this scalar product is equivalent to the usual norm on $\mathbb{S}^2_1(\mathbb{R}^N)$.

3.1.2. Existence and uniqueness of integral solutions

Lemma 3.4. *The conjugate operator of the operator A is the operator A^* defined as follows*

$$A^* = -\begin{pmatrix} 0 & I \\ P_{\alpha,\beta} - \gamma(X)I & 0 \end{pmatrix}$$

with

$$D(A^*) = \left\{ \begin{pmatrix} \chi \\ \psi \end{pmatrix} : \chi, \psi \in \mathbb{S}^2_1(\mathbb{R}^N), P_{\alpha,\beta}\chi - \gamma(X)\chi \in L^2(\mathbb{R}^N) \right\}.$$

From this lemma and Stone's Theorem, there exists a C_0-semi-group e^{tA} on H.

Lemma 3.5. *Suppose that $f(X,\xi)$ satisfies the condition ii). Then the following statements are true.*

a) The Nemytskii mapping

$$\begin{aligned}\widehat{f}:\mathbb{S}_1^2(\mathbb{R}^N) &\longrightarrow L^2(\mathbb{R}^N)\\ u &\longmapsto \widehat{f}(u)(X):=f(X,u(X))\end{aligned}$$

is Lipschitzian on every bounded set of $\mathbb{S}_1^2(\mathbb{R}^N)$.

b) The mapping

$$\begin{aligned}f^*:H &\longrightarrow H\\ U &\longmapsto f^*(U):=\begin{pmatrix}0\\ -\lambda v(X)+f(X,u(X))\end{pmatrix}\end{aligned}$$

is Lipschitzian on every bounded set of H.

We will now convert Problem (3.1)-(3.2) to a Cauchy problem for a system of first order equations. Let $v(X,t)=u_t(X,t)$ and

$$U=\begin{pmatrix}u(X,t)\\ v(_X,t)\end{pmatrix},\ A=\begin{pmatrix}0 & I\\ P_{\alpha,\beta}-\gamma(X)I & 0\end{pmatrix},$$
$$f^*(U)(X,t)=\begin{pmatrix}0\\ -\lambda v(X,t)+f(X,u(X,t))\end{pmatrix},\ U_0=\begin{pmatrix}u_0(X)\\ u_1(X)\end{pmatrix},$$

where I is the unitary operator on $S_1^2(\mathbb{R}^N)$. Then Problem (3.1)-(3.2) is equivalent to the following problem for a system of equations of the first order

$$\frac{\mathrm{d}U}{\mathrm{d}t}=AU+f^*(U), \tag{3.9}$$

$$U(0)=U_0. \tag{3.10}$$

Let $H=\mathbb{S}_1^2(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$, then H is a Hilbert space with the inner product

$$(U,\overline{U})_H=\left(\begin{pmatrix}u\\ v\end{pmatrix},\begin{pmatrix}\overline{u}\\ \overline{v}\end{pmatrix}\right)=((u,\overline{u}))_{\mathbb{S}_1^2(\mathbb{R}^N)}+(v,\overline{v})_{L^2(\mathbb{R}^N)}.$$

The domain $D(A)$ of operator A is given by

$$D(A)=\left\{\begin{pmatrix}u\\ v\end{pmatrix},u,v\in\mathbb{S}_1^2(\mathbb{R}^N);P_{\alpha,\beta}u-\gamma(X)u\in L^2(\mathbb{R}^N))\right\}.$$

Definition 3.1. Let $T>0,T\in\mathbb{R}$. A continuous mapping $U:[0,T)\to H$ is called *an integral solution* of Problem (3.9)-(3.10) if it is a solution of the integral equation

$$U(X,t)=e^{tA}U_0+\int_0^t e^{(t-s)A}f^*(U(X,s))\mathrm{d}s,\quad t\in[0,T).$$

If U is differentiable almost everywhere on $[0,T)$ together with U_t and AU belonging to $L^1_{loc}([0,T),H)$ and satisfies the differential equation

$$\frac{\mathrm{d}U}{\mathrm{d}t} = AU + f^*(U) \quad \text{almost everywhere in} \quad (0,T) \quad \text{and} \quad U(0) = U_0,$$

then U is called *a strong solution* of Problem (3.9)-(3.10).

Proposition 3.1. *Assume the conditions i) and ii) are satisfied. Then for any $R > 0$ there exists $T = T(R) > 0$ small enough such that for $U_0 \in H, \|U_0\|_H \leq R$, Problem* (3.9)-(3.10) *has one and only one integral solution that satisfies*

$$U(X,t) = e^{At}U_0 + \int\limits_0^t e^{A(t-s)} f^*(U(X,s))\mathrm{d}s, \quad t \in [0,T),$$

on the interval $[0,T)$. The solution can be expanded on $[0,\tau)$ and we have either $\tau = +\infty$ or $\lim\limits_{t\to\tau^-} \|U(.,t)\|_H = +\infty$.

Lemma 3.6. *Assume the conditions i) and ii) are satisfied. Then the solution $u(t)$ of Problem* (3.1)-(3.2) *satisfies*

$$\|u\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)} + \|u_t\|^2_{L^2(\mathbb{R}^N)} \leq M, \quad \forall t \geq 0, \tag{3.11}$$

where M is a constant that depends only on the data $\gamma(X), g(X), g_1(X), g_2(X)$ and *R when $\|u_0\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)} + \|u_1\|^2_{L^2(\mathbb{R}^N)} \leq R$.*

Theorem 3.1. Suppose that *i), ii) are satisfied and $U_0 \in H$. Then Problem* (3.1)-(3.2) *has a unique global solution $U \in C([0,\infty);H)$. Furthermore, for each fixed t the mapping $U_0 \mapsto S(t)U_0 := U(t)$ is continuous on H.*

3.2 The existence of a compact global attractor in $\mathbb{S}^2_1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$

Lemma 3.7. *Suppose that i), ii) are satisfied and $\mathcal{B}$ is the aforementioned bounded set in H. Then for each solution -$U(t) = (u(X,t), u_t(X,t))$* of Problem (3.9)-(3.10) *along with the original data $U_0 \in \mathcal{B}$ we have*

$$\lim_{T,R\to+\infty} \frac{1}{T} \int\limits_0^T \int\limits_{|X|_{\alpha,\beta}\geq R} \Big(|u(X,t)|^2 + |u_t(X,t)|^2 + |\nabla_{\alpha,\beta}u(X,t)|^2\Big)\mathrm{d}X\mathrm{d}t = 0, \tag{3.12}$$

where

$$|X|_{\alpha,\beta} = \Big[|x|^{2(1+\alpha+\beta)} + |y|^{2(1+\alpha+\beta)} + (1+\alpha+\beta)^2|z|^2\Big]^{\frac{1}{2(1+\alpha+\beta)}}.$$

Lemma 3.8. *Assume i), ii) and* $U_n \rightharpoonup U$ *in* H*. Then for each* $t \geq 0$ *we have*

$$S(t)U_n \rightharpoonup S(t)U \quad in \quad H. \tag{3.13}$$

Lemma 3.9. *Assume i), ii) and let* $\{U_n\}_{n=1}^{\infty}$ *be weakly convergent to* U *in* H*. Then*

$$\lim_{T\to+\infty} \limsup_{n\to+\infty} \|S(T)U_n - S(T)U\|_H = 0. \tag{3.14}$$

We present the main results of this chapter as follows.

Theorem 3.2. *Assume i), ii). Then* $\{S(t)\}_{t\geq 0}$ *is asymptotically compact in* H*, that is, for all bounded sequences* $\{U_n\}_{n=1}^{\infty}$ in H *and any non-negative sequence* $\{t_n\}_{n=1}^{\infty}$ *satisfying* $t_n \to +\infty$ when $n \to +\infty$, *then from* $\{S(t_n)U_n\}_{n=1}^{\infty}$ *we can extract a subsequence that converges in* H.

The following theorem confirms the existence of a global attractor of semi-group $S(t)$ in H.

Theorem 3.3. *Assume i), ii). Then the semi-group* $S(t)$ *generated by Problem* (3.9)-(3.10) *has a global attractor* $\mathcal{A}_H$ in H.

We consider the dynamic system $(H, S(t))$ which was described in Section 3.2. We denote by $\mathcal{N}_0$ the set of stationary states of this system, i. e.

$$\mathcal{N}_0 = \{(u,0) \in H := -P_{\alpha,\beta}u + \gamma(X)u = f(X,u), X \in \mathbb{R}^N\}.$$

Assume U(t) is an orbit. Then U(t) is a full orbit if

$$\lim_{i\to-\infty} \text{dist}(\text{U(t)}, \mathcal{N}_0) = \lim_{i\to+\infty} \text{dist}(U(t), \mathcal{N}_9) = 0.$$

The unstable manifold of this dynamic system will be determined by

$$W^u(\mathcal{N}_0) = \bigcup W^u(u_0, 0),$$

where $W^u(u_0,0) = \{(u,v); S(t)(u_0,0) \to (u,v)$ when $t \to +\infty\}$. Below is a theorem on the existence of a compact and connected global attractor together with the description of its structure.

Theorem 3.4. *Assume i) and ii). Then the dynamic system* $(H, S(t))$ *is associated with Problem* (3.1)-(3.2) *is a gradient and asymptotically compact system, and at the same time there is a connected compact global attractor* $\mathcal{A}$*. Furthermore,* $\mathcal{A} = W^u(\mathcal{N}_0)$ *and* $\mathcal{A}$ *is the unstable manifold of this dynamic system.*

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

The main results achieved in the thesis include:

1. For fourth order semi-linear equations containing the Δ_γ^2-Laplace: Give the Pohozaev integral identity of the Dirichlet problem, thereby prove the non-existence of non-trivial solutions in δ_t-star domains when the right hand sides grow, with respect to u, greater than $\frac{\widetilde{N}+4}{\widetilde{N}-4}$; at the same time, prove the existence of weak solutions, establish multiple solutions of the problem provided that the nonlinear terms grow, in terms of u, less than $\frac{\widetilde{N}+4}{\widetilde{N}-4}$. Thereby, it has been shown that the value $\frac{\widetilde{N}+4}{\widetilde{N}-4}$ is the critical value of the growth order of the right-hand sides in terms of u.
2. For hyperbolic equations containing the strongly degenerate elliptic operator $P_{\alpha,\beta}$: Give sufficient conditions for the linear and nonlinear terms of the equation to ensure the unique existence of the global integral solution; the existence of a connected compact global attractor in the space $S_1^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, and describe its structure.

Recommendations for further research

Regarding the topic of the thesis, the following issues are open and in our opinion are of interest:

1. Study conditions for existence of solutions of the above boundary problems in unbounded domains.
2. Study the properties of global attractors such as: dimensionality, continuous dependence on parameters, smoothness, ...
3. Study the existence of solutions and asymptotic behavior of degenerate hyperbolic equations with different boundary conditions, such as heterogeneous boundary conditions, Neumann boundary conditions, mixed boundary conditions, nonlinear boundary conditions, ... To do this, it is necessary to construct the corresponding weighted spaces, Sobolev type embedding theorems.

4. Study the existence of regressive and uniformly attractors when the nonlinear terms depend on the time.
5. Practical application models.

List of published works related to the thesis

[1] D. T. Luyen and P. T. K. Yen (2021), "Long time behavior of solutions to semi-linear hyperbolic equations involving strongly degenerate elliptic differential operators", *J. Korean Math. Soc.* 58 (2021), No. 5, 1279–1298.

[2] D. T. Luyen., H. T. Ngoan and P. T. K. Yen (2022), "Existence and non-existence of solutions for semilinear bi-Δ_γ-Laplace equation". *Bull. Malays. Math. Sci. Soc.* 45, No. 2, 819–838.