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ON THE NEVANLINNA THEORY FOR ANNULI AND UNIQUENESS PROBLEM

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INTRODUCTION

1. History and reasons for choosing the topic

Originated by the research results of R.Nevanlinna from the beginning of the XX century. Value distribution theory for meromorphic functions (also called Nevanlinna Theory) is regarded as one of the profound and beautiful achievements of Mathematics. The main content include two fundamental theorems: First Main Theorem and Second Main Theorem, Value distribution theory, increasingly attracting the attention of many domestic and foreign authors, obtained many important results and have applications in many different fields of Mathematics such as the uniqueness problem for meromorphic functions, complex dynamic systems, complex differential equations,....

Let $\mathbb{P}^n(\mathbb{C})$ be the complex projective space of complex dimension n over $\mathbb{C}$. In 1933, H. Cartan extended the Nevanlinna's second main theorem to the case for holomorphic curves sharing hyperplanes in general position into complex projective spaces and give some application. In this direction of research, many domestic and foreign mathematicians have published many outstanding results about the forms of The First Main Theorem and Second Main Theorem in different cases and to study the application of these theorems in different areas of Mathematics. Specially is the uniqueness problem for holomorphic curves.

Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map where $(f_0, \dots, f_n)$ be a reduced representative of f. The function

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta$$

is called *the Nevanlinna-Cartan characteristic function* of f, where $\|f(z)\| = \max\{|f_0(z)|, \dots, |f_n(z)|\}$.

Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ and let L be a linear form defined in H. The function

$$m_f(r,H)=m_f(r,L):=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|f(re^{i\theta})\|}{|L(f)(re^{i\theta})|}d\theta$$

is called *the proximity function* of f with respect to H. Let $n_f(r,H)$ be the number of zeros of $L(f)(z)$ in the disk $\{|z|<r\}$, counting multiplicity, and $n_f^M(r,H)$ be the number of zeros of $L(f)(z)$ in the disk $\{|z|<r\}$, multiple truncated by a positive integer M. The function

$$N_f(r,H)=N_f(r,L)=\int_0^r\frac{n_f(t,H)-n_f(0,H)}{t}dt+n_f(0,H)\log r$$

is called *the counting function* and the function

$$N_f^M(r,H)=N_f^M(r,L)=\int_0^r\frac{n_f^M(t,H)-n_f^M(0,H)}{t}dt+n_f^M(0,H)\log r$$

is called *the truncated counting function* by M of f with respect to H, where $n_f(0,H)=\lim\limits_{r\to 0}n_f(r,H)$, $n_f^M(0,H)=\lim\limits_{r\to 0}n_f^M(r,H)$. Number M denote by $N_f^M(r,H)$ is called *the multiplicity truncated.*

In 1933, H. Cartan proved the following:

Theorem 1. *Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ and let $f:\mathbb{C}\to\mathbb{P}^n(\mathbb{C})$ be a holomorphic curve whose image is not contained in H. We have*

$$T_f(r)=N_f(r,H)+m_f(r,H)+O(1).$$

Theorem 2. *Let $f:\mathbb{C}\to\mathbb{P}^n(\mathbb{C})$be a linearly non-degenerate holomorphic curve and let $H_1,\dots,H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in the general position, Thus, we get*

$$(q-n-1)T_f(r)\leqslant\sum_{j=1}^{q}N_f^n(r,H_j)+o(T_f(r))$$

holds for $r>0$ outside a set of finite Lebesgue measure

Theorem 1 is the first main theorem, and Theorem 2 is the second main theorem with truncated multiplicities for linearly non-degenerate holomorphic curves from $\mathbb{C}$ to $\mathbb{P}^n(\mathbb{C})$ intersecting a collection of fixed hyperplanes in general position. The results of this study by H. Cartan is considered very important, it opens a new research direction in developing the Value distribution theory - study the distribution of values meromorphic maps, holomorphic maps - which we know today by the name associated with two outstanding mathematicians

"**Nevanlinna-Cartan Theory**". Recent research results in this direction is focus on two issues:

1. Construct the fundamental theorems (the first main theorem and the second main theorem) for holomorphic curves form $\mathbb{C}$ or a domain in $\mathbb{C}$ to $\mathbb{P}^n(\mathbb{C})$ or a algebraic variety map in $\mathbb{P}^n(\mathbb{C})$ with the target of the hyperplanes, fixed hypersurfaces or moving by establishing a relationship between the Nevanlinna-Cartan characteristic with the proximity functions, the counting functions or the truncated counting functions. From which to infer results about defect relation.

2. Research the applications of Nevanlinna-Cartan theory in different areas of mathematics, such as research degeneracy of algebraic curves, uniqueness problem for meromorphic function and holomorphic curves, system of differential equations, complex partial derivatives,....

The first research direction has attracted the attention of many mathematicians and obtained many profound results, such as G. Dethloff, E. I. Nochka, M. Ru, P. Vojta, H. H. Khoai, D. D. Thai, T. V. Tan, T. T. H. An, S. D. Quang .... In 1983, Nochka extended the results of H. Cartan in the case of hyperplanes in the $N-$subgeneral position in $\mathbb{P}^n(\mathbb{C})$. In 2004, M. Ru established the second main theorem for holomorphic curves with target in the form of hypersurfaces in the general position in $(\mathbb{C})$ without ramification. In 2009, he extended that result for holomorphic curves to a projective algebraic manifold V. In 2007, T. T. H. An and H. T. Phuong and in 2008, Q. M. Yan and Z. H. Chen proved a relationship between the characteristic function $T_f(r)$ of holomorphic curves $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ with truncated counting functions $N_f^M(r, D_j)$ in the case encountering hypersurfaces in general position in projective space. In addition, in recent years, G. Dethloff, T. V. Tan, D. D. Thai, S. D. Quang, L. Shi, P. C. Hu, N. V. Thin ... many results have been published concerning the holomorphic curve of one or more complex variables to $\mathbb{P}^n(\mathbb{C})$ or a projective algebraic manifold in $\mathbb{P}^n(\mathbb{C})$ with target of hyperplanes or hypersurfaces, fied or moving, in in general position or $N-$subgeneral position.

One of the important applications of the Nevanlinna-Cartan Theory as well as the Nevanlinna Theory is to study about the determination of holomorphic maps as well as meromorphic function through the inverse image of one or more finite element sets. This problem also attracted the interest of many mathematicians: A. Boutabaa, W. Cherry, G. Dethloff, H. Fujimoto, M. Ru, L. Smiley, C. C. Yang, H. H. Khoai, D. D. Thai, T. V. Tan, S. D. Quang, H. T. Phuong and many other authors.

Let $f : U \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map, and $(f_0, \dots, f_n)$ be a reduced representative of f, where U be a complex plane $\mathbb{C}$ or a domain in $\mathbb{C}$. A collection of hypersurfaces $\mathcal{D} = \{D_1, \dots, D_q\}$, with any $D_j \in \mathcal{D}$, we denote

$$\begin{aligned}
&\overline{E}_f(D_j) = \{z \in U \mid Q_j \circ f(z) = 0 \ \text{ ignoring multiplicity}\};\\
&E_f(D_j) = \{(z, m) \in U \times \mathbb{N} \mid Q_j \circ f(z) = 0 \ \text{ and } \operatorname{ord}_{Q \circ f}(z) = m\}.
\end{aligned}$$

and set

$$\overline{E}_f(\mathcal{D}) = \bigcup_{D_j \in \mathcal{D}} \overline{E}_f(D_j) \quad \text{and} \quad E_f(\mathcal{D}) = \bigcup_{D_j \in \mathcal{D}} E_f(D_j).$$

Let $\mathcal{F}$ be a family of holomorphic maps form U to $\mathbb{P}^n(\mathbb{C})$. A collection hypersurfaces $\mathcal{D}$ is said to be *unique range set ignoring multiplicity*, denoted by URSIM (or *unique range set counting multiplicity,* denoted by URSCM) for a family of holomorphic maps $\mathcal{F}$ if for any pair of holomorphic maps $f, g \in \mathcal{F}$, the condition $\overline{E}_f(\mathcal{D}) = \overline{E}_g(\mathcal{D})$ (resp. $E_f(\mathcal{D}) = E_g(\mathcal{D})$) implies $f \equiv g$. The URSIM, URSCM are called *the unique range set* for a family of holomorphic maps $\mathcal{F}$ to the same.

In 1975, H. Fujimoto proved a result about uniqueness problem for meromorphic maps to complex projective space, shows that there exist unique range set counting multiplicity with $3n + 2$ hyperplanes in general position which determines a meromorphic maps. This result is considered as the opening for research on the uniqueness problem for meromorphic maps. Since that time, this proble has been studied intensively and many mathematicians expanded the Fujimoto's result. Following this result, in 1983, L. Smiley introduced a new result on uniqueness problem for linearly non-degenerate meromorphic maps by the inverse image of a finite family of hyperplanes, this problem was studied by H. Fujimoto again in 1998. In 2006, Dethloff and Tan considered the same problem for the moving hyperplanes. And many more results about uniqueness problem for holomorphic curves in the case of multiple variables published by M. Ru, D. D. Thai, T. V. Tan, D. Quang .... Note that, most proofs of the results about unique range sets are based on forms of the Second Main Theorem with truncated multiplicity.

For the uniqueness problem for meromorphic function, In 1926, R. Nevanlinna proved: *two nonconstant complex meromorphic function* f, g *satisfy* $f^{-1}(a_i) = g^{-1}(a_i),\ i = 1, \ldots, 5,$ *then* $f \equiv g$. This result of Nevanlinna shows that two meromorphic functions are uniquely determined by the inverse image of five distinct points. Following Nevanlinna's result. There are many published works of domestic and foreign authors, focusing on the following directions: meromorphic functions share a element, counting multiplicities and ignoring multiplicity.

Let f be a meromorphic function, denote

$$\sigma_2(f) = \liminf_{r \to \infty} \frac{\log \log T(r, f)}{\log r}.$$

Let f, g be two nonconstant meromorphic functions on complex plane and $a \in \mathbb{C}$. We say that f and g share the value a CM (counting multiplicities) if $f - a$ and $g - a$ have the same zeros, f and g share the value a CM if $f - a$ and $g - a$ have the same zeros counting multiplicities. In 1996, Brück posed the following conjecture which we later used to call the Brück conjecture: *Let* f *be a nonconstant entire function such that the hyper-order* $\sigma_2(f)$ *of* f *is not positive integer or* ∞. *If* f *and* f' *share a finite value* $a-$ *CM, then* $\dfrac{f' - a}{f - a} = c$, *where* c *is a nonzero constant.* Note that, The conjecture in the case $a = 0$ has been proved by Brück. In 1998, Gundersen and Yang verified that the Conjecture is true when f is of finite order (not an integer). In the case of f, a function of order with $\sigma_2(f) < \frac{1}{2}$ has been proved by Chen và Shon. However, the conjecture in the case $\sigma_2(f) \geq \frac{1}{2}$ is still open. It is intersting to ask what happens if f is replace by f^n, and f is replace by a differential polynomial or a is replace by a polynomial or a function. In 2008, L. Z. Yang and J. L. Zhang found out a result relating to Brück's conjecture as following: *Letf be a nonconstant entire function,* $n \geqslant 7$ *be an interger, and* $F = f^n$. *If* F *and* F' *share 1 CM, then* $F \equiv F'$ *and* f

assumes the form $f = ce^{z/n}$, where c is a nonzero constant. In 2008, Li and Cao extension of Brück conjecture when replacing the constant a by a suitable polynomial and replace the first derivative f' by higher order derivatives. These results create a new research direction, often called *A uniqueness problem for entire functions related to Brück conjecture.*

Thus, the further development of Nevanlinna-Cartan theory, especially study the form of truncated Second Main Theorem is really necessary. It will give us important basis for studying the uniqueness problem for meromorphic function and holomorphic maps. Currently, the problem of developing Nevanlinna-Cartan theory and research on the application of this theory as well as Nevanlinna theory in different sciences has been strongly interested, associated with the works of many domestic and foreign mathematicians: A. Boutabaa, H. Cartan, W. Cherry, G. Dethloff, Ph. Griffiths, M. Ru, P. Vojta, P. M. Wong, H. H. Khoai, D. D. Thai, T. T. H. An, S. D. Quang, H. T. Phuong, V. H. An and many other authors.

Choosing the topic **"On the Nevanlinna theory for annuli and uniqueness problem"** of author of this thesis also aims to continue to develop more interesting things of the Nevanlinna - Cartan Theory for holomorphic curves on annuli and uniqueness problem.

2. Purpose and research object

- **research object**

  In this thesis, we focus on studying the properties of meromorphic functions on the complex plane $\mathbb{C}$ and holomorphic curves on the annuli. These are also the basic research objects of Nevanlinna and Nevanlinna-Cartan theory.

- **Purpose:**

  The first direction of research: construct some form fundamental theorem (First Main Theorem and Second Main Theorem) for holomorphic curve on the annuli with target hypersurfaces by setting relate between Nevanlinna-Cartan characteristic function with proximity function, counting function or truncated counting function.

The second direction of research: setting some sufficient condition for two holomorphic curves on annuli sharing sufficiently many hypersurfaces in general position for Veronese embedding.

The third direction of research: construct some new results about uniqueness problem for meromorphic functions related to Brück conjecture in the case replace f by F and replace f' by $M[f]$.

3. Overview of the thesis

Throughout this thesis we denote $R > 1$ be a fixed positive real number or ∞ and

$$\Delta = \{z \in \mathbb{C} : \frac{1}{R} < |z| < R\}$$

be a annulus in $\mathbb{C}$.

One of the research directions in Nevanlinna and Nevanlinna-Cartan theory is to consider first main theorem and second main theorem for the mapping case from $f : \Delta \to \mathbb{P}^n(\mathbb{C})$. In the direction of this research, R. Korhonen, A. Khrystiyanyn and A. Kondratyuk have published the first publications about Value distribution for meromorphic functions on anuli. This problem immediately attracted the attention of authors around the world such as H. Cao, S. Liu, N. Lu, M. E. Lund, D. Meng and obtained some important results. For holomorphic curves on anuli, recently, in 2005, H. T. Phuong and N. V. Thin published some fundamental theorems for holomorphic curves on the annuli with target being fixed hyperplanes. The results that we achieve in this thesis about distribution of values for holomorphic curves on the annuli are the first main theorem and the second main theorem with target hypersurfaces. The specific results are as follows:

Theorem 1.2.3. *Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d and $f = (f_0 : \cdots : f_n) : \Delta \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve whose image is not contained D. Then, we have for any $1 < r < R$:*

$$m_f(r, D) + N_f(r, D) = dT_f(r) + O(1).$$

Theorem 1.3.6. *Let $f : \Delta \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate holomorphic curve, and let $D_j, 1 \leq j \leq q$, be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d_j in general position. Let d be the least common multiple of the $d_1, d_2, \ldots, d_q$. Let $0 < \varepsilon < 1$ and*

$$\alpha \geq (d[(n+1)^2 2^n])\varepsilon^{-1}] + 1)^n,$$

Then for any $1 < r < R$, we have

$$|| \quad (q - (n+1) - \varepsilon) T_f(r) \leqslant \sum_{j=1}^{q} d_j^{-1} N_f^{\alpha}(r, D_j) + O_f(r)$$

where

$$O_f(r) = \begin{cases} O(\log r + \log T_f(r)) & nu \quad R = +\infty \\ O(\log \dfrac{1}{R-r} + \log T_f(r)) & nu \quad R < +\infty. \end{cases}$$

Theorem 1.2.3 is a form of first main theorem for holomorphic functions on annuli. Theorem 1.3.6 is a form of truncated second main theorem for holomorphic functions on annuli from Δ to $\mathbb{P}^n(\mathbb{C})$ share hypersurfaces in general position in $\mathbb{P}^n(\mathbb{C})$, show a relate between the characteristic function $T_f(r)$ of holomorphic curves $f : \Delta \to \mathbb{P}^n(\mathbb{C})$ with the truncated counting functions $N_f^M(r, D_j)$.

For on uniqueness problem for holomorphic functions on annuli, in 2003, H. T. Phuong and T. H. Minh proved some results about uniqueness problem for holomorphic functions on annuli sharing fixed hyperplanes, năm 2021, H. H. Giang Giang published some results in this direction with the target of hyperplanes.... The results that we have obtained in this direction of research are as follows:

Theorem 2.2.1. *Let f and g be algebraically non-degenerate holomorphic curves from Δ into $\mathbb{P}^n(\mathbb{C})$ such that $O_f(r) = o(T_f(r))$ and $O_g(r) = o(T_g(r))$. Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of $q > n_{\mathcal{D}} + 1 + 2n_{\mathcal{D}}^2/m_{\mathcal{D}}$ hypersurfaces in general position for Veronese embedding in $\mathbb{P}^n(\mathbb{C})$ such that $f(z) = g(z)$ for all $z \in \overline{E}_f(\mathcal{D}) \cup \overline{E}_g(\mathcal{D})$. Then $f \equiv g$.*

Theorem 2.2.2. *Let f and g be algebraically non-degenerate holomorphic curves from Δ into $\mathbb{P}^n(\mathbb{C})$ such that $O_f(r) = o(T_f(r))$ and $O_g(r) = o(T_g(r))$. Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of $q > n_{\mathcal{D}} + 1 + 2n_{\mathcal{D}}/m_{\mathcal{D}}$ hypersurfaces in general position for Veronese embedding in $\mathbb{P}^n(\mathbb{C})$ such that*

(a) $f(z) = g(z)$ *for all* $z \in \overline{E}_f(\mathcal{D}) \cup \overline{E}_g(\mathcal{D})$,

(b) $\overline{E}_f(D_i) \cap \overline{E}_f(D_j) = \emptyset$ *and* $\overline{E}_g(D_i) \cap \overline{E}_g(D_j) = \emptyset$ *for all* $i \neq j \in \{1, \ldots, q\}$.

Then $f \equiv g$.

Theorem 2.2.1 and Theorem 2.2.2 are two algebraically conditions to uniquely identify holomorphic curve on annulus sharing sufficiently many hypersurfaces in general position for Veronese embedding.

Let f and g be two nonconstant meromorphic functions. Let a and b be two complex numbers. If $g - b = 0$ whenever $f - a = 0$, we write $f = a \Rightarrow g = b$. If $f = a \Rightarrow g = b$ and $g = b \Rightarrow f = a$, we write $f = a \Leftrightarrow g = b$. If $f - a$ and $g - b$ have the same zeros and poles (counting multiplicity), then we denote $f - a \rightleftharpoons g - b$. Base on the third direction of research concerning uniqueness problem for entire functions related to the Brück conjecture, we obtained the following result in 2018:

Theorem 3.2.4 *Let* $n \in \mathbb{N}$ *and* $k, n_i, t_i \in \mathbb{N}^*, i = 1, \ldots, k$ *satisfy one of the following conditions:*

$$1)\ k = 1,\ n = 0,\ n_1 \geqslant t_1 + 1;$$

$$2)\ n \geqslant 1 \text{ or } k \geqslant 2,\ n_j \geqslant t_j,\ n + \sum_{j=1}^{k} n_j \geqslant \sum_{j=1}^{k} t_j + 2.$$

Let a *and* b *be two*

nite nonzero values and f *be a nonconstant entire function. If* $f^{n+n_1+\cdots+n_k} = a \rightleftharpoons f^n(f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)} = b$ *then*

$$\frac{f^n(f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)} - b}{f^{n+n_1+\cdots+n_k} - a} = c,$$

where c *is a nonzero constant. Specially, if* $a = b$ *then* $f = c_1 e^{tz}$, *where* c_1 *and* t *are nonzero constants and* t *is satisfied by* $(tn_1)^{t_1} \ldots (tn_k)^{t_k} = 1$.

In order to prove Theorem 3.2.4, we need to use the following result about normal family of meromorphic functions.

Theorem 3.2.3. *Let* $\mathcal{F}$ *be a family of meromorphic functions in a complex domain* D. *Let* a *and* b*be two complex numbers such that* $b \neq 0$, *let* $n \in \mathbb{N}$,

$k \in \mathbb{N}^*$ and n_j, t_j, $j = 1, 2, \ldots, k$ satisfy

$$n_j \geqslant t_j, \quad n + \sum_{j=1}^{k} n_j \geqslant \sum_{j=1}^{k} t_j + 3, \tag{1}$$

and $f^{n+n_1+\cdots+n_k} = a \Leftrightarrow f^n(f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)} = b$ for all $f \in \mathcal{F}$. Then $\mathcal{F}$ is a normal family. Furthermore, if $\mathcal{F}$ is a family of holomorphic functions, then the statement holds when (1) is replaced by one of the following conditions:

$$k = 1,\ n = 0,\ n_1 \geqslant t_1 + 1;$$

$$n \geqslant 1 \ \text{or}\ k \geqslant 2, n_j \geqslant t_j, n + \sum_{j=1}^{k} n_j \geqslant \sum_{j=1}^{k} t_j + 2.$$

For the proof technique we mainly use Theorem 3.2.4 combine it with normal family theory and the Nevanlinna theory.

4. Research Methods of the thesis

In this thesis, we use basic research methods: on the basis of a research-oriented literature review, we discover open problems that need to be solved and use our knowledge, techniques of complex analysis, Nevanlinna and Nevanlinna-Cartan value distribution theory, algebra geometry, normal family theory to suggest suitable methods or use some technique to solve the problems posed. In addition to publication in journals, The main results of the thesis have been reported at:

- Seminar of the Department of Calculus, Faculty of Mathematics, University of Education, Thai Nguyen University every year.
- International Conference on Algebra - Number Theory - Geometry - Topology 2021, 21 - 23 October 2021 at University of Education, Thai Nguyen University.

CHAPTER 1

TWO FUNDAMENTAL THEORY FOR HOLOMORPHIC CURVES ON THE ANNULI

1.1. Some basic knowledge in the value distribution theory for meromorphic functions

Let f be a meromorphic function on complex plane $\mathbb{C}$. The function

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f(re^{i\varphi})\right|d\varphi$$

is called *the proximity function* of f. Let $n(r,f)$ be the number of poles of f in $\overline{D}_r=\{z\in\mathbb{C}:|z|\leqslant r\}$. The function

$$N(r,f)=\int_0^r\frac{n(t,f)-n(0,f)}{t}dt+n(0,f)\log r$$

is called *the truncated counting function* of f. The function $T(r,f)=m(r,f)+N(r,f)$ is called *the characteristic function* of f.

Theorem 1.1.1 (The first main theorem)**.** *Let f be a nonconstant meromorphic function on $\mathbb{C}$ and a be a complex number. Then*

$$T(r,\frac{1}{f-a})=T(r,f)+O(1).$$

Theorem 1.1.2 (The second main theorem)**.** *Let f be a nonconstant meromorphic function on $\mathbb{C}$. Let $a_1,\dots,a_q$ be q any distinct points in $\mathbb{C}$. Then*

$$(q-1)T(r,f)\leq\overline{N}(r,f)+\sum_{i=1}^{q}\overline{N}(r,\frac{1}{f-a_i})+S(r,f)$$

hols for $r\in[1,\infty)$ outside a set of finite Lebesgue measure, where $S(r,f)=o(T(r,f))$ when $r\to\infty$.

Let $R > 1$ be a fixed positive real number or $+\infty$, set:

$$\Delta = \Big\{z \in \mathbb{C} : \frac{1}{R} < |z| < R\Big\},$$

be a annulus in $\mathbb{C}$, and for any real number r, such that $1 < r < R$, we denote:

$$\Delta_r = \{z \in \mathbb{C} : \frac{1}{r} < |z| < r\} = \Delta_{1,r} \cup \Delta_{2,r}.$$

Let $f(z)$ be a meromorphic function on Δ and z_0 is a point of Δ. If $f(z)$ be the number of zeros having multiplicity α at z_0 then we denote $\nu_f(z_0) = \alpha$. If $f(z)$ be the number of poles having multiplicity α at z_0 then we denote $\nu_{f,\infty}(z_0) = \alpha$. In the case $f(z_0) \neq 0, \infty$, we denote $\nu_f(z_0) = 0$ and $\nu_{f,\infty}(z_0) = 0$. Let f be a meromorphic function on Δ, namely f holomorphic on Δ outside some abnormal point of poles, we recall

$$m(r, f) = m(r, \infty) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|d\theta,$$

where $a \in \mathbb{C}$ and $r \in (R^{-1}, R)$ and $\log^+ x = \max\{0, \log x\}$ with real numbers $x > 0$. With some real numbers $r \in \{1 < r < R\}$, we denote $m_0(r, f) = m(r, f) + m(r^{-1}, f)$. The function $m_0(r, f)$ is called *the proximity function* or *the compensation function* of f at $a \in \mathbb{C}$ and at ∞. Let $n_1(r, f)$ be the number of poles counting multiplicity of f in $\Delta_{1,r}$, $n_2(r, f)$ be the number of poles counting multiplicity of f in $\Delta_{2,r}$, namely

$$n_1(r, f) = \sum_{z\in\Delta_{1,r}} \nu_{f,\infty}(z), \quad n_2(r, f) = \sum_{z\in\Delta_{2,r}} \nu_{f,\infty}(z).$$

We set

$$N_1(r, f) = \int_{1/r}^{1} \frac{n_1(t, f)}{t}dt; \quad N_2(r, f) = \int_{1}^{r} \frac{n_2(t, f)}{t}dt.$$

The counting function at poles counting multiplicity of f is defined by $N_0(r, f) = N_1(r, f)+N_2(r, f)$. The characteristic function $T_0(r, f)$ of f is defined by $T_0(r, f) = m_0(r, f) - 2m(1, f) + N_0(r, f)$. The proximity functions, the counting functions and The characteristic functions are called Nevanlinna functions of meromorphic function on anuli.

The following lemmas is often referred to as the first main theorem and the Second Main Theorem for meromorphic functions on annuli:

Lemma 1.1.3. *Let f be a meromorphic function on Δ. Then for any $r \in (1, R)$, we have*

$$T_0(r, \frac{1}{f-c}) = T_0(r, f) + O(1)$$

holds for every fixed $c \in \mathbb{C}$.

Lemma 1.1.4. *Let f be a meromorphic function on $\Delta, a_1, a_2, \dots, a_p$ be distinct finite complex numbers and $\lambda \geqslant 0$. Then*

$$m_0(r, f) + \sum_{\nu=1}^{p} m_0\left(r, \frac{1}{f - a_\nu}\right) \leqslant 2T_0(r, f) - N_0^{(1)}(r, f) + S(r, f)$$

where

$$N_0^{(1)}(r, f) = N_0(r, 1/f') + 2N_0(r, f) - N_0(r, f')$$

and

i) in the case $R = \infty$ then $S(r, f) = O(\log(rT_0(r, f)))$ for $r \in (1, R)$, except for the set Δ_r such that $\int_{\Delta_r} r^{\lambda-1} dr < +\infty$;

ii) if $R < +\infty$ then $S(r, f) = O\left(\log\left(\frac{T_0(r, f)}{R-r}\right)\right)$ for $r \in (1, R)$, except for the set $\Delta_r^{'}$ such that $\int_{\Delta_r'} \frac{d_r)}{(R-r)^{\lambda-1}} < +\infty$.

1.2. Nevanlinna-Cartan functions and the first main theorem

Definition 1.2.1. Let *a holomorphic map form Δ to $\mathbb{P}^n(\mathbb{C})$, or also called holomorphic curves*, in projective space $\mathbb{P}^n(\mathbb{C})$ is defined as a mapping

$$\begin{aligned} f = (f_0 : \dots : f_n) : &\Delta \longrightarrow \mathbb{P}^n(\mathbb{C}) \\ &z \longmapsto (f_0(z) : \dots : f_n(z)), \end{aligned}$$

where $f_j, 0 \leqslant j \leqslant n$ be a entire functions on Δ. If $f_j, j = 0, 1, \dots, n$, be polynomials then f is called *algebraic curves* . In the case $f_0, \dots, f_n$ without common zeros on Δ then $(f_0, f_1, \dots, f_n)$ is called the reduced representation of f.

Definition 1.2.2. The holomorphic curves $f : \Delta \longrightarrow \mathbb{P}^n(\mathbb{C})$ is called *linearly degenerate* if the image of f is contained in some real linear manifold of projective space $\mathbb{P}^n(\mathbb{C})$. The holomorphic curves f is called *algebracally degenerate* if the image of f is contained in some real algebraic manifold of $\mathbb{P}^n(\mathbb{C})$.

Let $f = (f_0 : \cdots : f_n) : \Delta \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map where $f_0, \ldots, f_n$ are entire functions and without common zeros in Δ. For $1 < r < R$, characteristic function$T_f(r)$ of f is defined by:

$$T_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log \|f(r^{-1}e^{i\theta})\| d\theta,$$

where $\|f(z)\| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}$. The above definition is independent, up to an additive constant, of the choice of the reduced representation of f.

Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d, and let $Q.$ be the homogeneous polynomial of degree d defining D. The proximity function of f is defined by:

$$m_f(r, D) = \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\|f(re^{i\theta})\|^d}{|Q \circ f(re^{i\theta})|} d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\|f(r^{-1}e^{i\theta})\|^d}{|Q \circ f(r^{-1}e^{i\theta})|} d\theta.$$

Let M be a positive integer. Let $n_{1,f}(r, D)$ be the number of zeros of $Q(f)$ in $\Delta_{1,r}$ counting multiplicity, $n_{2,f}(r, D)$ being the number of zeros of $Q(f)$ in $\Delta_{2,r}$ counting multiplicity, $n^M_{1,f}(r, D)$ be the number of zeros of $Q(f)$ in $\Delta_{1,r}$, where any zero of multiplicity greater than M is "truncated" and counted as if it only had multiplicity M and $n^M_{2,f}(r, D)$ being the number of zeros of $Q(f)$ in $\Delta_{2,r}$, where any zero of multiplicity greater than M is "truncated" and counted as if it only had multiplicity M. We have:

$$n_{1,f}(r, D) = n_1(r, 1/Q(f)), \quad n_{2,f}(r, D) = n_2(r, 1/Q(f));$$
$$n^M_{1,f}(r, D) = n^M_1(r, 1/Q(f)), \quad n^M_{2,f}(r, D) = n^M_2(r, 1/Q(f)).$$

We denote:

$$N_{1,f}(r, D) = \int_{r^{-1}}^1 \frac{n_{1,f}(t, D)}{t} dt, \qquad N_{2,f}(r, D) = \int_1^r \frac{n_{2,f}(t, D)}{t} dt,$$
$$N^M_{1,f}(r, D) = \int_{r^{-1}}^1 \frac{n^M_{1,f}(t, D)}{t} dt, \qquad N^M_{2,f}(r, D) = \int_1^r \frac{n^M_{2,f}(t, D)}{t} dt.$$

The integrated counting and truncated functions are defined by:

$$N_f(r, D) = N_f(r, Q) := N_{1,f}(r, D) + N_{2,f}(r, D),$$
$$N^M_f(r, D) = N^M_f(r, Q) := N^M_{1,f}(r, D) + N^M_{2,f}(r, D).$$

Let $D_1,\dots,D_q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ and Q_j, $1\leqslant j\leqslant q$, be homogeneous polynomials in $\mathbb{C}[z_0,\dots,z_n]$ defining d_j. We recall that hypersurfaces $D_1,\dots,D_q$ are said to be in general position if $q>n$ and for any distinct $i_1,\dots,i_{n+1}\in\{1,\dots,q\}$. We have $\{z\in\mathbb{P}^n(\mathbb{C}):Q_{i_1}(z)=Q_{i_2}(z)=\cdots=Q_{i_{n+1}}(z)=0\}=\emptyset$.

In 2022, we proved the following theorems

Theorem 1.2.3. *Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d and $f=(f_0:\cdots:f_n):\Delta\to\mathbb{P}^n(\mathbb{C})$ be a holomorphic curve whose image is not contained D. Then, we have for any $1<r<R$:*

$$m_f(r,D)+N_f(r,D)=dT_f(r)+O(1).$$

Theorem 1.2.3 gives us an equality related between the characteristic function of an holomorphic curve on an annuli with proximity functions and counting functions combined with hypersurface. This result is similar to the case of holomorphic curves on the complex plane $\mathbb{C}$.

1.3. The second main theorem

We have proved the following theorem in 2022 is the second main theorem for holomorphic curves from Δ to $\mathbb{P}^n(\mathbb{C})$ intersecting a collection of fixed hypersurfaces in general position with truncated counting functions.

Theorem 1.3.1. *Let $f:\Delta\to\mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate holomorphic curve, and let $D_j, 1\leq j\leq q$, be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d_j in general position. Let d be the least common multiple of the $d_1,d_2,\dots,d_q$. Let $0<\varepsilon<1$ and*

$$\alpha\geq(d[(n+1)^2 2^n])\varepsilon^{-1}]+1)^n,$$

Then for any $1<r<R$, we have

$$||\quad(q-(n+1)-\varepsilon)T_f(r)\leqslant\sum_{j=1}^{q}d_j^{-1}N_f^{\alpha}(r,D_j)+O_f(r)$$

where

$$O_f(r)=\begin{cases}O(\log r+\log T_f(r)) & nu\quad R=+\infty\\ O(\log\dfrac{1}{R-r}+\log T_f(r)) & nu\quad R<+\infty.\end{cases}$$

CHAPTER 2

THE UNIQUENESS PROBLEM FOR HOLOMORPHIC CURVES ON ANNULI

2.1. Some preliminaries

In this section, we recall the first main theorem and second main theorems proved by H. T. Phuong and N. V. Thin in 2015, which are necessary for proofs of our results.

Lemma 2.1.1. *Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ and $f : \Delta \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve whose image is not contained H. Then we have for any $1 < r < R$,*

$$T_f(r) = m_f(r, H) + N_f(r, H) + O(1),$$

where $O(1)$ is a constant independent of r.

Lemma 2.1.2. *Let $f : \Delta \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic curve and $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Then we have*

$$|| \quad (q - n - 1)T_f(r) \leqslant \sum_{j=1}^{q} N_f^n(r, H_j) + O_f(r),$$

where

$$O_f(r) = \begin{cases} O(\log r + \log T_f(r)) & nu \quad R_0 = +\infty \\ O(\log \dfrac{1}{R_0 - r} + \log T_f(r)) & nu \quad R_0 < +\infty. \end{cases}$$

2.2. Two fundamental theorems for holomorphic curves

Let D be a hypersurface of degree d in $\mathbb{P}^n(\mathbb{C})$, which is defined by a homogeneous polynomial Q of degree d. Then

$$Q(z_0, \ldots, z_n) = \sum_{k=0}^{n_d} a_k z_0^{i_{k0}} \ldots z_n^{i_{kn}},$$

where $n_d = \binom{n+d}{n} - 1$ and $i_{k0} + \cdots + i_{kn} = d$, $a_k \in \mathbb{C}$ vi $k = 1, \ldots, n_d$. We denote by $\mathbf{a} = (a_0, \ldots, a_{n_d})$ the vector associated with D (or with Q).

Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of arbitrary hypersurfaces and D_j be the homogeneous polynomial in $\mathbb{C}[z_0, \ldots, z_n]$ of degree d_j defining D_j for $j = 1, \ldots, q$. Let $m_{\mathcal{D}}$ s the least common multiple of the d_j for $j = 1, \ldots, q$ and denote

$$n_{\mathcal{D}} = \binom{n + m_{\mathcal{D}}}{n} - 1.$$

We set $Q_j^* = Q_j^{m_{\mathcal{D}}/d_j}$ and let $\mathbf{a}_j^*$ be the vector associated with Q_j^* for $j = 1, \ldots, q$. The collection $\mathcal{D}$ is said to be in general position for Veronese embedding if $q > n_{\mathcal{D}}$ and for any distinct $i_1, \ldots, i_{n_{\mathcal{D}}+1} \in \{1, \ldots, q\}$ the vectors $\mathbf{a}_{i_1}^*, \ldots, \mathbf{a}_{i_{n_{\mathcal{D}}+1}}^*$ are linearly independent.

Let $f : \Delta \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve, let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d and Q be the homogeneous polynomial of degree d in $n+1$ variables with coefficients in $\mathbb{C}$ defining D, we define

$$\overline{E}_f(D) := \{z \in \Delta \mid Q \circ f(z) = 0 \ \text{ ignoring multiplicity}\};$$
$$E_f(D) := \{(z, m) \in \Delta \times \mathbb{N} \mid Q \circ f(z) = 0 \ \text{ and } \nu_{Q(f)}(z) = m \geqslant 1\}.$$

Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ in $\mathbb{P}^n(\mathbb{C})$ be a collection of hypersurfaces, we define

$$\overline{E}_f(\mathcal{D}) := \bigcup_{D_j \in \mathcal{D}} \overline{E}_f(D_j) \quad \text{and} \quad E_f(\mathcal{D}) := \bigcup_{D_j \in \mathcal{D}} E_f(D_j).$$

With the above concepts and notations, in 2021 we have proved a uniqueness theorem for meromorphic curves on an annuli sharing hypersurfaces as follows:

Theorem 2.2.1. *Let f and g be algebraically non-degenerate holomorphic curves from Δ into $\mathbb{P}^n(\mathbb{C})$ such that $O_f(r) = o(T_f(r))$ and $O_g(r) = o(T_g(r))$. Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of $q > n_{\mathcal{D}} + 1 + 2n_{\mathcal{D}}^2/m_{\mathcal{D}}$ hypersurfaces in general position for Veronese embedding in $\mathbb{P}^n(\mathbb{C})$ such that $f(z) = g(z)$ for all $z \in \overline{E}_f(\mathcal{D}) \cup \overline{E}_g(\mathcal{D})$. Then $f \equiv g$.*

Theorem 2.2.2. *Let f and g be algebraically non-degenerate holomorphic curves from Δ into $\mathbb{P}^n(\mathbb{C})$ such that $O_f(r) = o(T_f(r))$ and $O_g(r) = o(T_g(r))$. Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of $q > n_{\mathcal{D}} + 1 + 2n_{\mathcal{D}}/m_{\mathcal{D}}$ hypersurfaces in general position for Veronese embedding in $\mathbb{P}^n(\mathbb{C})$ such that*

(a) *$f(z) = g(z)$ for all $z \in \overline{E}_f(\mathcal{D}) \cup \overline{E}_g(\mathcal{D})$,*

(b) *$\overline{E}_f(D_i) \cap \overline{E}_f(D_j) = \emptyset$ and $\overline{E}_g(D_i) \cap \overline{E}_g(D_j) = \emptyset$ for all $i \neq j \in \{1, \ldots, q\}$.*

Then $f \equiv g$.

Some remark. 1. In theorem 2.2.2, the minimum number of hypersurfaces satisfying the hypothesis is

$$n_{\mathcal{D}} + 1 + 2n_{\mathcal{D}}/m_{\mathcal{D}}.$$

Note that if hypersurfaces in general position for Veronese embedding in $\mathbb{P}^n(\mathbb{C})$. In this case $n_{\mathcal{D}} = n$ and $m_{\mathcal{D}} = 1$ then $q = 3n + 2$, coincides with the required hyperplane number in Fujimoto's result.

2. We know that, holomorphic function h is said to be transcendental function if $\limsup\limits_{r \to \infty} \dfrac{T_0(r, h)}{\log r} = \infty$ in the case $R = \infty$ and $\limsup\limits_{r \to R} \dfrac{T_0(r, h)}{-\log(R - r)} = \infty$ in the case $R < \infty$. So for holomorphic curve f with reduced representative $f = (f_0 : \cdots : f_n)$, if one of f_j, is transcendental function then $O_f(r) = o(T_f(r))$.

CHAPTER 3

A UNIQUENESS PROBLEM FOR ENTIRE FUNCTIONS RELATED TO BRÜCK'S CONJECTURE

3.1. Preliminaries

Let a nonconstant meromorphic function $g(z)$ in the complex plane $\mathbb{C}$, and its first p derivatives. A dierential polynomial P of g is de

ned by

$$P(z) := \sum_{i=1}^{n} \alpha_i(z) \prod_{j=0}^{p} (g^{(j)}(z))^{S_{ij}},$$

where $S_{ij},\ 0 \leqslant i, j \leqslant n$, are nonnegative integers, and $\alpha_i(z), 1 \leqslant i \leqslant n$ are small meromorphic functions with respect to g. Set

$$d(P) := \min_{1 \leqslant i \leqslant n} \sum_{j=0}^{p} S_{ij} \ \text{ và } \ \theta(P) := \max_{1 \leqslant i \leqslant n} \sum_{j=0}^{p} j S_{ij}.$$

In 2002, J. Hinchliffe proved the following result, for evaluation between the Nevanlinna functions with the meromorphic functions and the counting functions with the differential polynomials.

Lemma 3.1.1. *Let g be a transcendental meromorphic function and $a \neq 0$ be a nonzero complex constant, let P be a nonconstant dierential polynomial in g with $d(P) \geqslant 2$. Then*

$$T(r, g) \leqslant \frac{\theta(P)+1}{d(P)-1} \overline{N}(r, \frac{1}{g}) + \frac{1}{d(P)-1} \overline{N}(r, \frac{1}{P-a}) + o(T(r, g)),$$

for all $r \in [1, +\infty)$ excluding a set of finite Lebesgues measure. When f is a *transcendental entire function, the above inequality becomes*

$$T(r,g) \leqslant \frac{\theta(P)+1}{d(P)}\overline{N}(r,\frac{1}{g}) + \frac{1}{d(P)}\overline{N}(r,\frac{1}{P-a}) + o(T(r,g)),$$

for all $r \in [1,+\infty)$ *excluding a set of finite Lebesgues measure.*

Let f a meromorphic function in the complex plane $\mathbb{C}$, we recall *the order* $\sigma(f)$ of meromorphic function f is defined by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

and *the order* of f is defined by

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

Special case, if f be an entire function, we know that f ca be expressed by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then we denote by

$$\mu(r,f) = \max_{n\in\mathbb{N},|z|=r} \{|a_n z^n|\},$$
$$\nu(r,f) = \sup\{n : |a_n| r^n = \mu(r,f)\},$$
$$M(r,f) = \max_{|z|=r} |f(z)|.$$

In this case, the order of f can be expressed in the form

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log\log(M(r,f))}{\log r}.$$

Lemma 3.1.2. *Let* f *is an entire function with the order* $\sigma(f)$*, then*

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log \nu(r,f)}{\log r}.$$

Lemma 3.1.3. *Let* f *be a transcendental entire function,* δ *be a real number, satisfy* $0 < \delta < \frac{1}{4}$. *Let* z *be a complex number, satisfy* $|z| = r$ *and*

$$|f(z)| > M(r,f)\nu(r,f)^{-\frac{1}{4}+\delta}.$$

Then there exists a set $F \subset \mathbb{R}_+$ *of finite logarithmic measure, that is* $\int_F \frac{dt}{t} < +\infty$, *such that*

$$\frac{f^{(m)}(z)}{f(z)} = \left(\frac{\nu(r,f)}{z}\right)^m (1+o(1))$$

holds for all $m \geqslant 1$ *and* $r \notin F$.

Taking $E_0(z) = 1 - z$, $E_m(z) = (1-z)e^{z+z^2/2+\cdots+z^m/m}, m \in \mathbb{Z}^+$, then we have a following result called the Weierstrass Factorization Theorem.

Lemma 3.1.4. *Let* f *be an entire function, with a zero multiplicity* $m \geqslant 0$. *at* $z = 0$. *Let the other zeros of* f *be at* $a_1, a_2, \ldots,$ *each zero being repeated as many times as its multiplicity implies. Then* f *has the representation*

$$f(z) = e^{g(z)} z^m \prod_{n=1}^{\infty} E_{m_n}(\frac{z}{a_n}),$$

for some entire function g *and some integers* m_n. *Furthermore, if* f *has finite order* ρ, *then* g *is a polynomial with degree at most* ρ.

Let S be a the Riemann sphere and $\pi : \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \to S$ be a spherical projection. Let $z_1, z_2 \in \widehat{\mathbb{C}}$, denote $M_1 = \pi(z_1)$ and $M_2 = \pi(z_2)$ be the two points in S corresponding to z_1, z_2 respectively. The length of the line segment $\overline{M_1M_2}$ is called *the spherical distance* between two points z_1, z_2 and denote by $\rho_S(z_1, z_2)$.

Definition 3.1.5. Let $S = \{f_n(z), n = 1, 2, \ldots\}$ be a sequence of meromorphic functions in a domain D. Let E be a set of points of D. The sequence S is said to be uniformly convergent in E with respect to the spherical distance, if to each positive number ε corresponds a positive integer N such that, when $n \geq N, m \geq N$,we have $\rho_S(f_n(z), f_m(z)) < \varepsilon$ in $z \in E$.

Definition 3.1.6. Cho $D \subset \mathbb{C}$ là mt min và $\mathcal{F}$ là mt h các hàm phân hình trên D. H $\mathcal{F}$ c gi là *h chun tc* trên D nu mi dãy $\{f_n\} \subset \mathcal{F}$ luôn tn ti mt dãy con ca $\{f_n\}$ hi t cu u trên mi tp con compact ca D.

3.2. A uniqueness problem

In order to prove our results about normal families of meromorphic functions, we first prove some lemmas necessary below

Lemma 3.2.1. *Let* f *be a transcendental meromorphic function and* a *a be a* *nonzero complex constant. Let* $n \in \mathbb{N}$, $k, n_j, t_j \in \mathbb{N}^*$, $j = 1, \ldots, k$ *satisfy*

$$n + \sum_{j=1}^{k} n_j \geqslant \sum_{j=1}^{k} t_j + 3.$$

Then the equation

$$f^n (f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)} = a$$

has infinite solutions. Furthermore, if f *is a transcendental entire function, the statement holds when* $n + \sum\limits_{j=1}^{k} n_j \geqslant \sum\limits_{j=1}^{k} t_j + 2$.

Lemma 3.2.2. *Let* f *be a nonconstant rational function and* a *be a nonzero complex constant. Let* $n \in \mathbb{N}$, $k, n_j, t_j \in \mathbb{N}^*$, $j = 1, \ldots, k$ *satisfy*

$$n_j \geqslant t_j, \quad n + \sum_{j=1}^{k} n_j \geqslant \sum_{j=1}^{k} t_j + 2, \ j = 1, \ldots, k.$$

Then the equation

$$f^n (f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)} = a$$

has at least two distinct zeros.

Let f and g be two nonconstant meromorphic functions. Let a and b be two complex numbers. If $g - b = 0$ whenever $f - a = 0$, we write $f = a \Rightarrow g = b$. If $f = a \Rightarrow g = b$ and $g = b \Rightarrow f = a$, we write $f = a \Leftrightarrow g = b$. If $f - a$ and $g - b$ have the same zeros and poles (counting multiplicity), then we denote $f - a \rightleftharpoons g - b$. Using this concept we have proved the following result about a normal criteria for family of meromorphic functions.

Theorem 3.2.3. *Let* $\mathcal{F}$ *be a family of meromorphic functions in a complex domain* $D \subset \mathbb{C}$. *Let* a *and* b*be two complex numbers such that* $b \neq 0$, *let* $n \in \mathbb{N}$, $n_j, t_j, k \in \mathbb{N}^*$, $(j = 1, 2, \ldots, k)$ *satisfy*

$$n_j \geqslant t_j, n + \sum_{j=1}^{k} n_j \geqslant \sum_{j=1}^{k} t_j + 3 \tag{3.1}$$

and

$$f^{n+n_1+\cdots+n_k} = a \Leftrightarrow f^n (f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)} = b \tag{3.2}$$

for all $f \in \mathcal{F}$. Then $\mathcal{F}$ is a normal family. Furthermore, if $\mathcal{F}$ is a family of holomorphic functions, then the statement holds when (3.1) is replaced by one of the following conditions:

$$k = 1,\ n = 0,\ n_1 \geqslant t_1 + 1; \tag{3.3}$$

$$n \geqslant 1 \text{ or } k \geqslant 2, n_j \geqslant t_j, n + \sum_{j=1}^{k} n_j \geqslant \sum_{j=1}^{k} t_j + 2. \tag{3.4}$$

Let f be a meromorphic funtion, denote

$$M[f] := f^n (f^{n_1})^{(t_1)} \dots (f^{n_k})^{(t_k)} \text{ và } F = f^{n+n_1+\dots+n_k},$$

here $n, n_1, ..., n_k, t_1, ..., t_k$ are positive integers.
Our following theorem is a result about uniqueness problem for meromorphic functions related to Brück conjecture when f is replaced by F and f' by $M[f]$.

Theorem 3.2.4. *Let $n \in \mathbb{N}$ and $k, n_i, t_i \in \mathbb{N}^*, i = 1, \dots, k$ satisfy one of the following conditions:*

$$1)\ k = 1,\ n = 0,\ n_1 \geqslant t_1 + 1;$$

$$2)\ n \geqslant 1 \quad or \quad k \geqslant 2,\ n_j \geqslant t_j,\ n + \sum_{j=1}^{k} n_j \geqslant \sum_{j=1}^{k} t_j + 2.$$

Let a and b be two finite nonzero values and f be a nonconstant entire function. If $F = a \rightleftharpoons M[f] = b$, then

$$\frac{M[f] - b}{F - a} = c,$$

where c is a nonzero constant.Specially, if $a = b$ then $f = c_1 e^{tz}$, where c_1 and t are nonzero constants and t is satisfies by $(tn_1)^{t_1} \dots (tn_k)^{t_k} = 1$.

As a special case, if we take $n = 0,\ k = 1,\ t_1 = 1$ in Theorem 3.2.4, then we have:

Corollary 3.2.5. *Let f be a nonconstant entire function, $n \geqslant 2$ be an integer, and $F = f^n$. If F and F' share 1 CM, then $F \equiv F'$, and f assumes the form $f = ce^{z/n}$, where c is a nonzero constant.*

Note. As stated in the introduction, in 2008, L. Z. Yang and J. L. Zhang proved a similar result corollary 3.2.5 with condition $n \geqslant 7$. Thus Theorem 3.2.4 is an improvement of the result of Yang và Zhang.

CONCLUSION AND RECOMMENDATION

The thesis has studied some form the fundamental theorems in Nevanlinna - Cartan theory for holomorphic curves on annuli in the case of hypersurfaces and the uniqueness problem for holomorphic curves on annuli and the entire functions related to the Brück conjecture.

The thesis has obtained the following results:

1. The first and second fundamental theorems for holomorphic curves on the annuli in the target cases are hypersurfaces.

2. Two uniqueness theorems for algebraically non-degenerate holomorphic curves on the annulus sharing sufficiently many hypersurfaces in general position for Veronese embedding.

3. New normal criteria for family of meromorphic functions on the complex plane $\mathbb{C}$ and prove a result about uniqueness problem for entire functions related to Brück conjecture.

Recommendation:

Many issues in this direction are in need of research. In the immediate future, we are interested in the following issues, which are directly related to the results we have obtained:

1. Study some Second Main Theorem for holomorphic curves on the annuli into an algebraic variety in $\mathbb{P}^n(\mathbb{C})$ in the case of hyperplane or hypersurface.

2. Study uniqueness problem for holomorphic curves on an annulus in the case of hypersurfaces in general position.

3. Study uniqueness problem for functions or for holomorphic curves based on second fundamental theorem with the new counting function.

LIST OF THE SCIENTIFIC PUBLICATIONS RELATED OF THE THESIS

1. Thin N.V., Phuong H.T., Vilaisavanh L. (2018), ”A uniqueness problem for entire functions related to Bruck's conjecture”, *Math. Slovaca* 68, No. 4, 823–836.

2. Phuong H. T., Vilaisavanh L. (2021), ”Some uniqueness theorems for holomorphic curves on annulus sharing hypersurfaces”, *Complex variable and Elliptic Equations*, Vol. 66(1), pp.22-34.

3. Phuong H. T., Vilaisavanh L. (2022), ”On fundamental theorems for holomorphic curves on an annulus intersecting hypersurfaces”, *Bulletin of the Iranian Mathematical Society*, Vol. 48, pp. 151-163.