THAI NGUYEN UNIVERSITY

UNIVERSITY OF EDUCATION

VU THI THUY DUONG

THE REGULARITY AND THE DECAY RATES OF SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

Speciality: Mathematical Analysis

Code: 946 01 02

DISSERTATION SUMMARY

THAI NGUYEN 2021

The dissertation was finished at:

THAI NGUYEN UNIVERSITY - UNIVERSITY OF EDUCATION

Supervisor: Prof. Dr. Sc. Nguyen Minh Tri

Referee 1:...

Referee 2: ....

Referee 3:...

The dissertation will be defended in the university committee:

THAI NGUYEN UNIVERSITY - UNIVERSITY OF EDUCATION

At............,2021

The dissertation is publicly available at:

- The Viet Nam National Library.
- The Thai Nguyen University Learning Resource Center.
- The Library of Thai Nguyen University - University of Education.

Introduction

1. HISTORY AND SIGNIFICANCE OF THE PROBLEM

Classical partial differential equations have been developed and studied intensively since the beginning of the nineteenth century and represent the foundation of knowledge about waves, heat transfer, hydrodynamics and other physics problems. The study of those real problems has motivated mathematicians to explore and apply new methods in pure mathematical research to solve partial differential equation problems. This is a large topic that is closely related to other sciences such as physics, mechanics, chemistry, engineering science and has many applications to industrial problems. Although the theory of partial differential equations has undergone a great development in the twentieth century, there are still a number of problems that remain unsolved up to now, mainly related to global existence, uniqueness, the solution, the smoothness as well as the decay rate of solutions.

One of the most famous and interesting forms of partial differential equations of mathematicians is the nonlinear Parabolic equation. Referring to nonlinear forms of Parabolic equations, we cannot fail to mention one of the seven famous millennium problems, which is the Navier-Stokes system of equations. It is the equation that describes the motion of a fluid, such as the flow of an ocean, or the creation of a small whirlpool within the currents.

The question of the uniqueness and regularity of the Navier-Stokes equations remains one of the 18 open problems of this century. So far there is no solution for the uniqueness of the solution except for small time intervals, and it has been questioned whether the Navier-Stokes equations really describe general flows? However, they also cannot prove they are not unique. It is possible that the methods used so far are not suitable and the Navier-Stokes equation system needs a different approach.

The uniqueness of the solutions of the equations is the foundation of the study of motion problems in partial differential equations. If more than one solution satisfies the same initial condition, it is said that the space of solutions is too large. Solution uniqueness can be restored if nonphysical solutions are excluded. More precisely, a non-unique result would contradict the study of fluid mechanics problems, and the introduction of a more complex model to study the motion of viscous fluids is necessary. If the problem of uniqueness involves the predictive aspect of the theory, then the problem of the existence of a solution touches on the question of physical model self-consistency regarding the Navier-Stokes equations. If the solution does not exist, the theory makes no sense.

In the twentieth century, instead of explicit formulas in special cases, the problem of solutions to the Navier-Stokes equations was studied in their general form. This leads to the concept of weak solutions. However, for weak solutions, only the existence of solutions can be guaranteed. Another problem related to the Navier-Stokes system of equations that has also attracted the attention of scientists in recent years is the problem of the asymptotic shape of the solution as time approaches infinity. Because when we know the asymptotic shape of the solution, we can predict the development trend of the system in the future and from there make appropriate assessments and adjustments.

Because of the above reasons, we have chosen the research topic for our thesis as: ***"The regularity and the decay rates of solutions to the Navier-Stokes equations"***.

2. GOALS AND OBJECT OF THE THESIS

a. Goals of the thesis

• *Topic 1*: Study the boundary value problems of the Navier-Stokes equations in a general domain with the following aspects:

- The regularity of weak solutions.
- The decay rates of weak solutions.

• *Topic 2*: Study the Cauchy problems of the Navier-Stokes equations in the three-dimensional space with the following aspect:

- The decay rates of strong solutions.

b. Object of the thesis

- The subjects of the thesis are the boundary value problems and the Cauchy problems of the Navier-Stokes equations in a general domain and in three-dimensional space.

3. RESEARCH METHODS OF THE THESIS

• The regularity of weak solutions for the Navier-Stokes equations in a general domain is studied by using the theory of the existence of local strong solutions and the uniqueness of strong solutions in a general domain and semi-group estimates.

• To study the decay rates of the weak solution for the Navier-Stokes equations in a general domain, we use the theory of the uniqueness and the decay rates of the strong solutions in a general domain, the embedding theorems and some semi-group estimates.

• The decay rates of strong solutions for the Navier-Stokes equations in the three-dimensional space is studied by using the theorem about the existence of local strong solutions, uniqueness of strong solutions in $\mathbb{R}^3$, the decay rates of the global strong solutions when the initial value is small enough with some tools of harmonic analysis.

4. RESULTS OF THE THESIS

In the thesis we obtain the following main results:

• For the regularity of the weak solutions problem in a general domain, we prove that u is regular if the kinetic energy $\frac{1}{2}\left\|u(t)\right\|_2^2$ is left-side Hölder continuous with Hölder exponent $\dfrac{1}{2}$ and with a sufficiently small Hölder semi-norm.

• For the decay rates of the weak solutions problem in a general domain, we prove that the time decay rates of the weak solution u in the L^2-norm like ones of the solutions for the homogeneous Stokes system taking the same initial value in which the decay exponent is less than $\dfrac{3}{4}$. Moreover, we show that under some additive conditions on the initial value, then u coincides with the solution of the homogeneous Stokes system when time tends to infinity.

• For the decay rates of the strong solutions problem in the three-dimensional space, we prove that the time decay rates of u in the L^3-norm coincide with ones of the heat equation with the initial value $|u_0|$.

5. STRUCTURE OF THE THESIS

Together with the Introduction, Overview, Conclusion, Author's works related to the thesis that have been published and References, the thesis includes three chapters:

• Chapter 1: In this chapter, we present some preliminaries.

• Chapter 2: In this chapter, we present two results on the regularity and the decay rates of weak solutions for the Navier-Stokes equations in a general domain.

• Chapter 3: In this chapter, we present the decay rates of weak solutions for the Navier-Stokes equations in three-dimensional space.

Overview of the thesis

The results on properties of solutions such as the existence, the uniqueness, the regularity and the decay rates of solutions of the Navier-Stokes equations have been mentioned quite a lot in domestic and foreign mathematical publications in recent years.... However, the development of the above results for the case of unbounded domains is still a new research direction that requires new approaches and technical tools in the proof. In this thesis, we study two properties of the solution, there are the regularity and the decay rates of the solution to the Navier-Stokes equations in a general domain $\Omega \subseteq \mathbb{R}^3$.

The problem of regularity of weak solutions for the Navier-Stokes equations was first obtained in 1982 by authors L. Caffarelli, R. Kohn and L. Nirenberg and was extended in many publications of mathematicians in the world in recent years such as the authors H. Sohr, H. Kozono, R. Farwig, W. Varnhorn, PF Riechwald .... However, most of the results are only obtained for bounded domains in the three-dimensional space $\mathbb{R}^3$.

In 2008 and 2009, R. Farwig, H. Kozono, and H. Sohr obtained the same results about the regularity but the domain Ω is additionally supposed to be bounded. Consider the weak solutions of the instationary problem of the Navier-Stokes system

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u|_{\partial\Omega} = 0, \\ u(0, x) = u_0 \end{cases} \tag{0.1}$$

in a general domain $\Omega \subseteq \mathbb{R}^3$. They proved the regularity of u under a condition

$$\varliminf_{\delta \to 0^+} \frac{\left|\frac{1}{2} \left\|u(t_0 - \delta)\right\|_2^2 - \frac{1}{2} \left\|u(t_0)\right\|_2^2\right|}{\delta^\alpha} < \infty, \tag{0.2}$$

where $\frac{1}{2} < \alpha < 1$. In 2010, they improved the their results, in which the condition (0.2) is replaced by the weaker condition

$$\varliminf_{\delta \to 0^+} \frac{\left|\dfrac{1}{2} \left\|u(t_0 - \delta)\right\|_2^2 - \dfrac{1}{2} \left\|u(t_0)\right\|_2^2\right|}{\delta^{\frac{1}{2}}} < C$$

and the domain Ω is bounded.

In 2016, R. Farwig and P. F. Riechwald proved the regularity of weak solutions when Ω is a general unbounded domain with uniform C^2-boundary $\partial\Omega$. Since the usual Stokes operator A_q cannot be defined on all types of unbounded domains Farwig and Riechwald have to replace the space $L^q(\Omega), q > 2$ by $\tilde{L}^q(\Omega) = L^q(\Omega) \cap L^2(\Omega)$.

The space $\bar{L}^q$ is equipped with the norm

$$\|u\|_{\bar{L}^q} := \max\{\|u\|_q, \|u\|_2\} \text{ if } q \geq 2$$

and

$$\|u\|_{\bar{L}^q} := \inf\left\{\|u_1\|_q + \|u_2\|_2 : u = u_1 + u_2\right\} \text{ if } 1 \leq q < 2$$

where $u_1 \in L^q(\Omega), u_2 \in L^2(\Omega)$.

In 2012, R. Farwig, H. Sohr, and W. Varnhorn showed that if u is a weak solution of the Navier-Stokes system (3.1) satisfying $u \in L^\infty_{\text{loc}}\big([0,T), L^3(\Omega)\big)$ with a bounded domain Ω or $u \in L^\infty_{\text{loc}}\big([0,T), \mathbb{D}(A^{\frac{1}{4}})\big)$ with a general domain Ω then u satisfies the local right-hand side Serrin condition in $[0,T)$.

In the next section, we will introduce some results obtained with the problems of the decay rates of the solutions for the Navier-Stokes equations. The decay rates of the solution problem in $L^2(\Omega)$ for the Navier-Stokes equations was first studied in 1934 by J. Leray in the space $\mathbb{R}^3$. The first results about the decay rates of the solutions for the Navier-Stokes equations was proved by T. Kato in 1984 in the case $\Omega = \mathbb{R}^d$, $d = 3.4$. From his publication developed studies for strong solutions in the general L^p spaces. The result of M. E. Schonbek applied to the case where Ω is a half of $\mathbb{R}^d$, $d \geq 2$ or a domain of $\mathbb{R}^d$, $d \geq 3$.

In 1986, the authors R. Kajikiya, T. Miyakawa continued to develop results on the decay rates of weak solutions for the Navier-Stokes equations in $\mathbb{R}^d$, $d = 3.4$ with the initial value $u_0 \in L^2_\sigma(\mathbb{R}^d)$, the authors proved that there exists a weak solution u of the Navier-Stokes equations satisfy the following properties:

(i) $\|u(t)\|_2 \to 0$ when $t \to \infty$.

(ii) If $u_0 \in L^2_\sigma(\mathbb{R}^d) \cap L^r_\sigma(\mathbb{R}^d)$ with $1 \leqq r < 2$, then

$$\|u(t)\|_2 \leq Ct^{-\frac{\frac{d}{r} - \frac{d}{2}}{2}} \quad \text{for all } t > 0,$$

where C is is a positive constant that depends only on d, r and u_0.

In 1992, W. Borchers and T. Miyakawa improved the results in the case of the unbounded domain, they showed that if $\|e^{-tA}u_0\|_2 = O(t^{-\alpha})$ with $\alpha \in \left(0, \frac{1}{2}\right)$, then $\|u(t)\|_2 = O(t^{-\alpha})$.

Consider the Navier-Stokes equations in a general domain $\Omega \subset \mathbb{R}^3$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0 & (x \in \Omega, t > 0) \\ \text{div } u = 0 & (x \in \Omega, t \geqq 0) \\ u\big|_{\partial\Omega} = 0 & \\ u\big|_{t=0} = u_0, & \end{cases}$$

where $u = (u_1, u_2, u_3)$ and pressure p are unknown quantities. We have the main result of W. Borchers and T. Miyakawa as follows:

Let $\Omega \subset \mathbb{R}^3$ is any unbounded domain and $u_0 \in L^2_\sigma(\Omega)$. Then, there exists a weak solution u of the Navier-Stokes equations such that

(i) $\|u(t)\|_2 \to 0 \quad$ when $t \to \infty$.

(ii) If $\left\|e^{-tA}u_0\right\|_2 = O\left(t^{-\alpha}\right)$ with $\alpha > 0$, then

$$\|u(t)\|_2 = \begin{cases} O\left(t^{-\alpha}\right) & \text{if } \alpha < \dfrac{1}{2}, \\ O\left(t^{\varepsilon - \frac{1}{2}}\right) & \text{if } \alpha \geq \dfrac{1}{2}, \end{cases}$$

where $0 < \varepsilon < \dfrac{1}{2}$.

The second problem studied in the thesis is the Cauchy problem for the Navier-Stokes equations in three-dimensional space. Consider the following integral:

$$u(t) = S(t)u_0 - \int_0^t S(t-s)\mathbb{P}\nabla \cdot (u \otimes u)(s)\mathrm{d}s.$$

The global existence of weak solutions was studied for the first time by J. Leray in 1934 and E. Hopf in 1951. In 1964, the problem of the existence of global strong solutions of the Navier-Stokes equations with the value small enough in Sobolev space $\dot{H}^{1/2}$ was studied by H. Fujita and T. Kato, then developed by J. Y. Chemin in 2009.

In recent years, the problem of existence of a global solution has continued to be developed in other spaces such as Sobolev-Lorentz spaces, Sobolev-Fourier-Lorentz spaces, homogeneous Sobolev-Lorentz spaces and Besov spaces by authors N. M. Tri and D. Q. Khai from 2014 to 2017.

In 1984, T. Kato studied the decay rates of the solution in the spaces $L^q(\mathbb{R}^d)$ by applying the estimates $L^q - L^p$ to the semi-group which is generated by the Stokes operator. The author has shown that there exists $T > 0$ and a unique solution u satisfying

$$t^{\frac{1}{2}(1-\frac{d}{q})}u \in BC([0,T); L^q) \text{ with } d \leq q \leq \infty,$$

$$t^{\frac{1}{2}(2-\frac{d}{q})}\nabla u \in BC([0,T); L^q) \text{ with } d \leq q \leq \infty,$$

when $u_0 \in L^d(\mathbb{R}^d)$. Moreover, the above estimate is hold for $T = \infty$ if $\left\|u_0\right\|_{L^d(\mathbb{R}^d)}$ small enough.

In 1997, M. Cannone generalized the results of T. Kato. The author has shown that if $u_0 \in L^d$ and $\left\|u_0\right\|_{\dot{B}_q^{\frac{d}{q}-1,\infty}}, (q > d)$ small enough, then there exists a unique solution u satisfying

$$t^{\frac{1}{2}(1-\frac{d}{q})} u \in BC([0,\infty); L^q) \text{ with } q \geq d.$$

For the the decay rates problem of a strong solution in a large time interval, if $u \in C([0,\infty), X)$ is a global solution with $u_0 \in X$, where X is $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ or $L^3(\mathbb{R}^3)$, then we have $\lim\limits_{t\to\infty} \left\|u(t)\right\|_X = 0$. These results were proved by I. Gallagher with $X = \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and with $L^3(\ mathbbR^3)$. For the case $X = L^3(\mathbb{R}^3)$, I. Gallagher proved the following result: Suppose $u \in C_t\left(L^3(\mathbb{R}^3)\right)$ is a milk solution of the Navier-Stokes equations.

Consider the Cauchy problem for the Navier-Stokes equations in $\mathbb{R}^3$:

$$\begin{cases} u_t - \Delta u + \nabla \cdot (u \otimes u) + \nabla p = 0, \\ \text{div } u = 0, \\ u(0,x) = u_0. \end{cases}$$

Then, the global solution u of the Navier-Stokes equations decays to zero as the time t goes to infinity in the space $L^3(\mathbb{R}^3)$, that is

$$\lim_{t\to\infty} \|u(\cdot,t)\|_3 = 0$$

and this solution is stable, that is, there exists a positive constant $\varepsilon(u)$ such that if $\|v_0 - u_0\|_3 < \varepsilon(u)$, then the local solution $v \in C_t\left(L^3(\mathbb{R}^3)\right)$ of the Navier-Stokes equations is a global solution with

$$\sup_{t\geqslant 0} \|v(\cdot,t) - u(\cdot,t)\|_3 < C(u)\,\|v_0 - u_0\|_3\,.$$

From the above studying results, we see that there are still many open problems related to the regularity and the decay rates of the solutions for the Navier-Stokes equations in the general domain and in the three-dimensional space. Therefore, in this thesis, we will present the following open problems:

- Studying on the regularity of solutions for the initial boundary problem of the Navier-Stokes equations in the general domain in three-dimensional space.

- Studying the decay rates of the solution for the initial boundary problem of the Navier-Stokes equations in the general domain in three-dimensional space.

- Studying the decay rates for the Cauchy problem of the Navier-Stokes equations in three-dimensional space.

Chapter 1

Preliminaries

Chapter 1 is a preparatory chapter that presents the most basic knowledge and is used to demonstrate the studying results achieved in the following chapters of the thesis. Specifically, Section 1.1 presents the function spaces to be used and the necessary inequalities in those spaces. Section 1.2 presents the basic operators in the Navier-Stokes equations and the estimators related to Stokes operators and Stokes semi-group. Finally, Section 1.3 introduces the Navier-Stokes equations and the classes of solutions of the Navier-Stokes equations studied in the thesis, including: weak, strong and milk solutions.

Chapter 2

The regularity and the decay rates of solutions to the Navier-Stokes equations in the general domain

In this chapter, we study the regularity and the decay rates of weak solutions for the initial boundary problem of the Navier-Stokes equations in the general domain. Here, we obtain theorems on the regularity, the decay rates of weak solutions in the general domain $\Omega \subseteq \mathbb{R}^3$. The method of proving theorems is based on the theory of the existence of local strong and global strong solutions, some properties of the bilinear operator $B(u, v)$, the embedding theorem and a Stokes semi-group estimates.

The content of this chapter is based on articles [1], [2] in the List of published scientific publications related to the thesis.

2.1 The regularity of weak solutions to the Navier-Stokes equations in the general domain

2.1.1. The setting of the problem

We consider the instationary problem of the Navier-Stokes system

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla p = 0, \\ \text{div } u = 0, \\ u|_{\partial\Omega} = 0, \\ u(0, x) = u_0 \end{cases} \tag{2.1}$$

in a general domain $\Omega \subseteq \mathbb{R}^3$, i.e a nonempty connected open subset of $\mathbb{R}^3$, not necessarily bounded, with boundary $\partial\Omega$ and a time interval $[0, T), 0 < T \leq \infty$ and with the initial value u_0, where $u = (u_1, u_2, u_3)$; $u \cdot \nabla u = \text{div}(uu), uu = (u_i u_j)_{i,j=1}$, if div $u = 0$.

Now we recall the definitions of weak and strong solutions to (3.1).

Definition 2.1.1. Let $u_0 \in L^2_\sigma(\Omega)$.

1. A vector field

$$u \in L^\infty\big(0,T; L^2_\sigma(\Omega)\big) \cap L^2_{\mathrm{loc}}\big([0,T); W^{1,2}_{0,\sigma}(\Omega)\big) \tag{2.2}$$

is called a *weak solution* (in the sense of Leray-Hopf) of the Navier-Stokes system (3.1) with the initial value $u(0) = u_0$ if the relation

$$-\langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} - \langle uu, \nabla w\rangle_{\Omega,T} = \langle u_0, w\rangle_\Omega \tag{2.3}$$

is satisfied for all test functions $w \in C_0^\infty\big([0,T); C^\infty_{0,\sigma}(\Omega)\big)$, and additionally the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u(\tau)\|_2^2 \mathrm{d}\tau \le \frac{1}{2}\|u_0\|_2^2 \tag{2.4}$$

is satisfied for all $t \in [0,T)$.

2. A weak solution u is called a *strong solution* of the Navier-Stokes equation (3.1) if additionally local *Serrin's condition*

$$u \in L^s_{\mathrm{loc}}\big([0,T); L^q(\Omega)\big) \tag{2.5}$$

is satisfied with exponents $2 < s < \infty, 3 < q < \infty$, where $\dfrac{2}{s} + \dfrac{3}{q} = 1$.

As is well known, in the case the domain Ω is bounded, it is not difficult to prove the existence of a weak solution u as in Definition 2.1.1 which additionally satisfies the *strong energy inequality*

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t'}^t \|\nabla u(\tau)\|_2^2 \mathrm{d}\tau \le \frac{1}{2}\|u(t')\|_2^2 \tag{2.6}$$

for almost all $t' \in [0,T)$ and all $t \in [t',T)$. For further results in this context for unbounded domains.

A weak solution u is called *regular* in some interval $(a,b) \subseteq (0,T)$ if Serrin's condition

$$u \in L^s_{\mathrm{loc}}\big(a,b\,; L^q(\Omega)\big) \tag{2.7}$$

is satisfied where $2 < s < \infty$, $3 < q < \infty$, $\dfrac{2}{s} + \dfrac{3}{q} = 1$.

A time $t \in (0,T)$ is called a *regular point* of u if there exists an interval $(a,b) \subseteq (0,T)$ such that u is regular in (a,b) with $a < t < b$.

2.1.2. Properties of the bilinear operator $B(u,v)$ and the Stokes semi-group e^{-tA}

First, we define an auxiliary space $\mathcal{K}^{\bar{s}}_{\tilde{s},T}$ which is made up of the functions u such that

$$t^{\frac{\alpha}{2}} u \in BC\big([0,T); \mathbb{D}(A^{\frac{\bar{s}}{2}})\big)$$

and

$$\lim_{t\to 0} t^{\frac{\alpha}{2}} \big\|A^{\frac{\bar{s}}{2}} u(t)\big\|_2 = 0 \tag{2.8}$$

with $-1<\tilde{s}\le\overline{s}<\infty$, $\alpha=\overline{s}-\tilde{s}$. The auxiliary space $\mathcal{K}^{\overline{s}}_{\tilde{s},T}$ is equipped with the norm

$$\|u\|_{\mathcal{K}^{\overline{s}}_{\tilde{s},T}} := \sup_{0<t<T} t^{\frac{\alpha}{2}}\big\|A^{\frac{\overline{s}}{2}}u(t)\big\|_2. \tag{2.9}$$

In the case $\overline{s}=\tilde{s}$, it is also convenient to define the space $\mathcal{K}^{\overline{s}}_{\tilde{s},T}$ as the natural subspace $BC\big([0,T);\mathbb{D}(A^{\frac{\overline{s}}{2}})\big)$ with the additional condition that its elements $u(t,x)$ satisfy

$$\lim_{t\to 0}\big\|A^{\frac{\overline{s}}{2}}u(t)\big\|_2=0.$$

We define

$$\mathcal{G}^{\overline{s}}_{\tilde{s},T} := \mathcal{K}^{\overline{s}}_{\tilde{s},T}\cap L^\infty\big(0,T;L^2_\sigma(\Omega)\big)\cap L^4\big([0,T);W^{1,2}_{0,\sigma}(\Omega)\big).$$

Lemma 2.1.1. *Let E and F be two normed spaces such that $E\cap F$ is a Banach space with the norm $\|x\|_{E\cap F}:=\|x\|_E+\|x\|_F$. Assume that B is a bilinear operator from $(E\cap F)\times(E\cap F)$ to $E\cap F$ such that there exists a positive constant $\eta>0$ satisfying*

$$\|B(x,y)\|_E\le\eta\|x\|_E\|y\|_E,\ \textit{for all } x,y\in E\cap F,$$
$$\|B(x,y)\|_F\le\eta\|x\|_E\|y\|_F,\ \textit{for all } x,y\in E\cap F,$$
$$\|B(x,y)\|_F\le\eta\|x\|_F\|y\|_E,\ \textit{for all } x,y\in E\cap F.$$

Then for any fixed $y\in E\cap F$ such that $\|y\|_E<\frac{1}{4\eta}$, the equation $x=y-B(x,x)$ has a unique solution $\overline{x}\in E\cap F$ satisfying $\|\overline{x}\|_E<\frac{1}{2\eta}$.

In the following lemmas a particular attention will be devoted to the study of the bilinear operator $B(u,v)$ defined by

$$B(u,v)=A^{\frac{1}{2}}\int_0^t e^{-(t-\tau)A}A^{-\frac{1}{2}}\mathbb{P}(u\cdot\nabla v)\mathrm{d}\tau.$$

Lemma 2.1.2. *Let $s_1,s_2,s_3,\overline{s}$ and $T\in\mathbb{R}$ be such that*

$$-1<s_1,s_2\le 1,\ s_1+s_2>0,\ -1<s_3\le s_1+s_2-\frac{1}{2},\ \max\Big\{s_3,-\frac{1}{2}\Big\}\le\overline{s}<\frac{3}{2},\quad T>0.$$

Then the operator B is a bilinear operator from $\mathcal{G}^1_{s_1,T}\times\mathcal{G}^1_{s_2,T}$ to $\mathcal{K}^{\overline{s}}_{s_3,T}$ and the following inequality holds

$$\Big\|B(u,v)\Big\|_{\mathcal{K}^{\overline{s}}_{s_3,T}}\lesssim T^{\frac{s_1+s_2-s_3-1/2}{2}}\|u\|_{\mathcal{K}^1_{s_1,T}}\|v\|_{\mathcal{K}^1_{s_2,T}}. \tag{2.10}$$

Lemma 2.1.3. *Let p and $s\in\mathbb{R}$ be such that*

$$\frac{1}{2}\le s<1,\ \frac{2}{1+s}<p<\infty.$$

Then the operator B is a bilinear operator from

$$\Big(\mathcal{G}^1_{s,T}\cap L^p\big([0,T);\mathbb{D}(A^{\frac{1}{2}})\big)\Big)\times\Big(\mathcal{G}^1_{s,T}\cap L^p\big([0,T);\mathbb{D}(A^{\frac{1}{2}})\big)\Big)\ \textit{to}\ L^p([0,T);\mathbb{D}(A^{\frac{1}{2}}))$$

and the following inequalities holds

$$\Big\|B(u,v)\Big\|_{L^p\big([0,T);\mathbb{D}(A^{\frac{1}{2}})\big)} \lesssim T^{\frac{s-1/2}{2}}\|u\|_{\mathcal{K}^1_{s,T}}\|v\|_{L^p\big([0,T);\mathbb{D}(A^{\frac{1}{2}})\big)}$$

and

$$\Big\|B(u,v)\Big\|_{L^p\big([0,T);\mathbb{D}(A^{\frac{1}{2}})\big)} \lesssim T^{\frac{s-1/2}{2}}\|v\|_{\mathcal{K}^1_{s,T}}\|u\|_{L^p\big([0,T);\mathbb{D}(A^{\frac{1}{2}})\big)}.$$

Lemma 2.1.4. *Let*

$$E = \mathcal{K}^1_{\frac{1}{2},T},\ \ F = \mathcal{G}^1_{\frac{1}{2},T}\cap\mathcal{G}^1_{0,T},$$

where $0 < T \le \infty$. The space F is equipped with the norm

$$\|u\|_F := \|u\|_{\mathcal{K}^1_{\frac{1}{2},T}} + \|u\|_{\mathcal{K}^1_{0,T}} + \|u\|_{L^4\big([0,T);\mathbb{D}(A^{\frac{1}{2}})\big)} + \|u\|_{L^\infty\big([0,T);L^2_\sigma(\Omega)\big)}.$$

Then the operator B is a bilinear operator from $F\times F$ to F satisfying

$$\|B(u,v)\|_E \ \le\ \eta\|u\|_E\,\|v\|_E, \quad \forall u,v\in F, \tag{2.11}$$

$$\|B(u,v)\|_F \ \le\ \eta\|u\|_E\,\|v\|_F, \quad \forall u,v\in F, \tag{2.12}$$

$$\|B(u,v)\|_F \ \le\ \eta\|u\|_F\,\|v\|_E, \quad \forall u,v\in F, \tag{2.13}$$

where η is a positive constant independent of T.

Lemma 2.1.5.

(a) *If $u_0\in\mathbb{D}(A^{\frac{s}{2}})$, $0\le s<1$ then $e^{-tA}u_0\in\mathcal{K}^1_{s,\infty}$.*

(b) *If $u_0\in\mathbb{D}(A^{\frac{1}{4}})$ then $e^{-tA}u_0\in F$.*

2.1.3. The regularity of weak solutions to the Navier-Stokes equations in the general domain

In this section, we present two results on the regularity of weak solutions for the Navier-Stokes equations in the general domain $\Omega\subseteq\mathbb{R}^3$. The first results extend the results of the group of authors R. Farwig, H. Kozono, H. Sohr in 2010 with Ω as the bounded domain and extend the results of R. Farwig, P. F. Riechwald in 2016 with general domain but boundary $\partial\Omega$ belongs to class C^2. To prove the main result we need the theorem on the existence of the strong local and the global strong solutions in the following general domain.

Theorem 2.1.6. *Let $\Omega\subseteq\mathbb{R}^3$ be a general domain. Then*

(a) *There exists a positive constant D such that for all $u_0\in\mathbb{D}(A^{\frac{1}{4}})$ and $0<T\le\infty$ satisfying*

$$\sup_{0\le t\le T} t^{\frac{1}{4}}\big\|A^{\frac{1}{2}}e^{-tA}u_0\big\|_2 < D \tag{2.14}$$

the Navier-Stokes system (3.1) has a strong solution u in time interval $[0,T)$ with the following properties:

$$u \in L^4\big([0,T);\mathbb{D}(A^{\frac{1}{2}})\big) \tag{2.15}$$

and

$$(1+t)^{\frac{1}{4}}u \in BC([0,T);\mathbb{D}(A^{\frac{1}{4}})\big). \tag{2.16}$$

In particular, for arbitrary $u_0 \in \mathbb{D}(A^{\frac{1}{4}})$*, there exists* $T = T(u_0)$ *small enough such that the inequality* (2.14) *holds.*

(b) *Suppose* $u_0 \in \mathbb{D}(A^{\frac{s}{2}})$, $\frac{1}{2} \le s \le 1$. *Then the inequality* (2.14) *holds if*

$$T^{\frac{1}{2}(s-\frac{1}{2})}\big\|A^{\frac{s}{2}}u_0\big\|_2 < D. \tag{2.17}$$

From the condition of existence of the local strong solutions and the global strong solutions of the Navier-Stokes equations in the general domain, we obtain the following main results.

Theorem 2.1.7. *Let* $\Omega \subseteq \mathbb{R}^3$ *be a general domain. Then there exists a positive constant* C *such that if* u *is a weak solution of the Navier-Stokes system* (3.1) *on* $(0,T)$ *verifying the strong energy inequality* (2.6) *and at* $t_0 \in (0,T)$ *the kinetic energy satisfying*

$$\varliminf_{\delta\to 0^+} \frac{\left|\frac{1}{2}\|u(t_0-\delta)\|_2^2 - \frac{1}{2}\|u(t_0)\|_2^2\right|}{\delta^{\frac{1}{2}}} < C, \tag{2.18}$$

then u *is regular at* t_0.

Theorem 2.1.8. *Let* $\Omega \subseteq \mathbb{R}^3$ *be a general domain. Then there exists a positive constant* C *such that if* u *is a weak solution of the Navier-Stokes system* (3.1) *on* $(0,T)$ *satisfying* $u(t) \in \mathbb{D}(A^{\frac{1}{4}})$ *for all* $t \in [0,T)$ *and*

$$\varliminf_{\delta\to 0^+} \big\|A^{\frac{1}{4}}\big(u(t-\delta) - u(t)\big)\big\|_2 < C \textit{ for all } t \in (0,T) \tag{2.19}$$

then $u \in L^4_{\mathrm{loc}}([0,T);L^6(\Omega))$.

In Theorem 2.2.7, if the function u is left-continuous from $[0,T)$ to $\mathbb{D}(A^{\frac{1}{4}})$ then $\varliminf\limits_{\delta\to 0^+} \big\|A^{\frac{1}{4}}\big(u(t-\delta) - u(t)\big)\big\|_2 = 0$ for all $t \in [0,T)$. Therefore, the condition (2.19) holds.

2.2 The decay rates of weak solutions to the Navier-Stokes equations in the general domain

In the second part of this chapter, we study the decay rates of weak solution to the Navier-Stokes equations in the general domain with the norm in $L^2(\Omega)$.

2.2.1. Properties of Stokes operators in the general domain

We consider the in-stationary problem of the Navier-Stokes system

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla p = 0, \\ \text{div } u = 0, \\ u|_{\partial\Omega} = 0, \\ u(0, x) = u_0, \end{cases} \tag{2.20}$$

in a general domain $\Omega \subseteq \mathbb{R}^3$, i.e a non-empty connected open subset of $\mathbb{R}^3$, not necessarily bounded, with boundary $\partial\Omega$ and a time interval $[0, T), 0 < T \leq \infty$ and with the initial value u_0, where $u = (u_1, u_2, u_3)$; $u \cdot \nabla u = \text{div}(uu), uu = (u_i u_j)_{i,j=1}$, if div $u = 0$.

Let us construct a weak solution of the following integral equation

$$u(t) = e^{-tA}u_0 - \int_0^t A^{\frac{1}{2}} e^{-(t-\tau)A} A^{-\frac{1}{2}} \mathbb{P}(u \cdot \nabla u) \mathrm{d}\tau. \tag{2.21}$$

We know that

$$u \in L^\infty\big(0, T; L^2_\sigma(\Omega)\big) \cap L^2_{\text{loc}}\big([0, T); W^{1,2}_{0,\sigma}(\Omega)\big)$$

is a weak solution of the Navier-Stokes system (3.1) iff u satisfies the integral equation (2.21).

In order to prove the main theorems, we need the following lemmas.

Lemma 2.2.1. *Let $\gamma, \theta \in \mathbb{R}$ and $t > 0$, then*

(a) *If $\theta < 1$, then*

$$\int_0^{\frac{t}{2}} (t - \tau)^{-\gamma} \tau^{-\theta} \mathrm{d}\tau = K_1 t^{1-\gamma-\theta}$$

where $K_1 = \int_0^{\frac{1}{2}} (1 - \tau)^{-\gamma} \tau^{-\theta} \mathrm{d}\tau < \infty$.

(b) *If $\gamma < 1$, then*

$$\int_{\frac{t}{2}}^t (t - \tau)^{-\gamma} \tau^{-\theta} \mathrm{d}\tau = K_2 t^{1-\gamma-\theta}$$

where $K_2 = \int_{\frac{1}{2}}^1 (1 - \tau)^{-\gamma} \tau^{-\theta} \mathrm{d}\tau < \infty$.

The proof of this lemma is elementary and may be omitted. □

Lemma 2.2.2. *Let $u \in L^2(\Omega)$ and $\nabla u \in L^2(\Omega)$. Then*

$$\left\| e^{-tA} \mathbb{P}(u \cdot \nabla u) \right\|_2 \lesssim t^{-\frac{\beta}{2}} \|u\|_2^{\beta - \frac{1}{2}} \|\nabla u\|_2^{\frac{5}{2} - \beta}$$

where β is positive constant such that $\dfrac{1}{2} \leq \beta < \dfrac{3}{2}$.

Lemma 2.2.3. *There exists a positive constant δ such that if $u_0 \in \mathbb{D}(A^{\frac{1}{4}})$ and $\|A^{\frac{1}{4}} u_0\|_2 \leq \delta$, then the Navier-Stokes system (3.1) has a strong solution with the initial value u_0 satisfying $\|\nabla u(t)\|_2 \lesssim t^{-\frac{1}{2}}$ for all $t \geq 0$.*

Lemma 2.2.4. *Let u be a weak solution of the Navier-Stokes system (3.1) with the initial value $u_0 \in L^2_\sigma(\Omega)$. Then there exists the positive value t_0 large enough such that $\|\nabla u(t)\|_2 \lesssim t^{-\frac{1}{2}}$ for all $t \geq t_0$.*

Lemma 2.2.5. *Let $u_0 \in L^2_\sigma(\Omega)$. Then*

(a) $\|e^{-tA}u_0\|_2 \to 0$ *as* $t \to \infty$.

(b) *If* $u_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)$ *for some* $1 < q \leq 2$, *then*

$$\left\|e^{-tA}u_0\right\|_2 = o\left(t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}\right) \text{ as } t \to \infty. \tag{2.22}$$

2.2.2. The decay rates of weak solutions to the Navier-Stokes equations in the general domain

The first results proved that the weak solution u of the Navier-Stokes equations with the norm in $L^2(\Omega)$ has the same decay rates in time as the solution of a homogeneous Stokes system with the same initial value and the exponent is less than $\dfrac{3}{4}$. This result extends the result of W. Borchers and T. Miyakawa with the convergence exponent $\alpha \in \left(0, \dfrac{1}{2}\right)$. The main results are as follows:

Theorem 2.2.6. *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain, $u_0 \in L^2_\sigma(\Omega)$ and u is a weak solution of the Navier-Stokes system and satisfying strong energy inequality. Then*

(a) *If* $\|e^{-tA}u_0\|_2 = O(t^{-\alpha})$ *for some* $0 \leq \alpha < \dfrac{3}{4}$, *then* $\|u(t)\|_2 = O(t^{-\alpha})$ *as* $t \to \infty$.

(b) *If* $\|e^{-tA}u_0\|_2 = o(t^{-\alpha})$ *for some* $0 \leq \alpha < \dfrac{3}{4}$, *then* $\|u(t)\|_2 = o(t^{-\alpha})$ *as* $t \to \infty$.

The second result shows that, when adding some conditions of the initial value, the weak solution u coincides with the solution of the homogeneous Stokes system as the time t goes to infinity. The results in Theorems 2.2.7 and Theorem 2.2.8 are stronger than those of W. Borchers and T. Miyakawa when the condition of the initial value is added.

Theorem 2.2.7. *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain, $u_0 \in L^2_\sigma(\Omega)$ and u is a weak solution of the Navier-Stokes system satisfying strong energy inequality. If $u_0 \in L^q(\Omega) \cap L^2_\sigma(\Omega)$, $1 < q \leq 2$, then*

$$\|u(t)\|_2 = o\left(t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}\right) \quad as \quad t \to \infty.$$

Theorem 2.2.8. *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain, $u_0 \in L^2_\sigma(\Omega)$ and u is a weak solution of the Navier-Stokes system satisfying strong energy inequality. If there exist positive constants t_0, C_1, and C_2 such that*

$$C_1 t^{-\alpha_1} \leq \|e^{-tA}u_0\|_2 \leq C_2 t^{-\alpha_2} \quad for \quad t \geq t_0,$$

where α_1, *and* α_2 *are constants satisfying*

$$0 \leq \alpha_2 < \frac{1}{2} \quad \textit{and} \quad \alpha_2 \leq \alpha_1 < \alpha_2 + \frac{1}{4},$$

then u *coincides with the solution of the homogeneous Stokes system with the initial value* u_0 *when time tends to infinity in the sense that*

$$\lim_{t\to\infty} \frac{\left\|u(t) - e^{-tA}u_0\right\|_2}{\|u(t)\|_2} = 0. \tag{2.23}$$

Chapter 3

The decay rates of strong solutions to the Navier-Stokes equations in the three-dimensional space

In this chapter, we study the decay rates of strong solutions to the Cauchy problem of the Navier-Stokes equations. Let $u \in C([0,\infty); L^3(\mathbb{R}^3))$ be a strong solution of the Cauchy problem for the Navier-Stokes equations with the initial value u_0 in the three-dimensional space. We prove that the time decay rates of u in the L^3-norm coincide with ones of the heat equation with the initial value $|u_0|$. Our proofs use the theory about the exitstence of local strong solutions, time decay rates of strong solutions when the initial value is small enough, and uniqueness arguments.

The content of this chapter is based on articles [3] in the List of published scientific publications related to the thesis.

3.1 Some properties of strong solutions to the Navier-Stokes equations in three-dimensional space

We consider the Cauchy problem of the incompressible Navier-Stokes equations in the whole space $\mathbb{R}^3$

$$\begin{cases} u_t - \Delta u + \nabla \cdot \big(u \otimes u\big) + \nabla p = 0, \\ \text{div } u = 0, \\ u(0,x) = u_0. \end{cases} \tag{3.1}$$

The unknown quantities are the velocity $u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))$ of the fluid element at time t and position x and the pressure $p(t,x)$.

In this section we prepare some auxiliary lemmas, we first establish the $L^p - L^q$ estimate for the heat semigroup with differential.

Lemma 3.1.1. *Assume that $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3, t > 0$ and $1 \leq p \leq q \leq \infty$. Then for all*

$f \in L^p$ we have

$$t^{\beta} D^{\alpha} e^{t\Delta} f \in BC([0,\infty); L^q(\mathbb{R}^3)) \text{ and } \left\| D^{\alpha} e^{t\Delta} f \right\|_q \leq C_{p,q,\alpha} t^{-\beta} \|f\|_p.$$

where $D^{\alpha} = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $\beta = \frac{3}{2}(\frac{1}{p} - \frac{1}{q}) + \frac{|\alpha|}{2}$, and $C_{p,q,\alpha}$ is a positive constant which depends only on p, q, and α.

Lemma 3.1.2. *For $t > 0$, the operator $O_t = e^{t\Delta}\mathbb{P}$ is a convolution operator $O_t f = K_t * f$, where the kernel K_t satisfies $K_t(x) = \frac{1}{t^{\frac{3}{2}}} K\left(\frac{x}{\sqrt{t}}\right)$ for a smooth function K such that*

$$(1 + |x|)^{3+|\alpha|} D^{\alpha} K \ \in L^{\infty}(\mathbb{R}^3),$$

where $|x| = \left(\sum_{i=1}^{3} x_i^2\right)^{1/2}$, $x = (x_1, x_2, x_3)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

Lemma 3.1.3. *Assume that $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, $t > 0$ and $1 \leq p < q \leq \infty$. Then for all $f \in L^p$ we have*

$$t^{\beta} D^{\alpha} e^{t\Delta} \mathbb{P} f \in BC([0,\infty); L^q(\mathbb{R}^3)) \text{ and } \left\| D^{\alpha} e^{t\Delta} \mathbb{P} f \right\|_q \leq C_{p,q,\alpha} t^{-\beta} \|f\|_p,$$

where $\beta = \frac{3}{2}(\frac{1}{p} - \frac{1}{q}) + \frac{|\alpha|}{2}$ and $C_{p,q,\alpha}$ is a positive constant which depends only on p, q, and α.

In order to proceed, we use the auxiliary space $\mathcal{K}_T^q$, $3 \leq q \leq \infty$ which is made up of the functions $u(t,x)$ such that

$$t^{\frac{1}{2}(1-\frac{3}{q})} u(t) \in BC\big([0,T); L^q(\mathbb{R}^3)\big)$$

and

$$\lim_{t \to 0} t^{\frac{1}{2}(1-\frac{3}{q})} \left\| u(t) \right\|_q = 0. \tag{3.2}$$

In the case $q = 3$, it is also convenient to define the space $\mathcal{K}^3$ as the natural subspace of $C([0,T); L^3(\mathbb{R}^3))$.

The space $\mathcal{K}_T^q$ is equipped with the norm

$$\left\| u \right\|_{\mathcal{K}_T^q} := \sup_{0 \leq t \leq T} t^{\frac{1}{2}(1-\frac{3}{q})} \left\| u(t,x) \right\|_q < \infty. \tag{3.3}$$

In the following lemmas a particular attention will be devoted to study of the bilinear operator $B(u,v)(t)$ defined by

$$B(u,v)(t) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot \big(u \otimes v\big) \mathrm{d}s.$$

Lemma 3.1.4. *The bilinear operator B is bicontinuous from $\mathcal{K}_T^q \times \mathcal{K}_T^q$ to $\mathcal{K}_T^p$ for any $3 \leq p < \dfrac{3q}{6-q}$ if $3 < q < 6$; any $3 \leq p < \infty$ if $q = 6$; and $\dfrac{q}{2} \leq p \leq \infty$ if $6 < q < \infty$ and the following inequality holds*

$$\|B(u,v)\|_{\mathcal{K}_T^p} \leq C \|u\|_{\mathcal{K}_T^q} \|v\|_{\mathcal{K}_T^q} \ \textit{ for all } u, v \in \mathcal{K}_T^q,$$

where C is a positive constant independent of T.

Lemma 3.1.5. *If $u_0 \in L^3(\mathbb{R}^3)$ then $e^{t\Delta}u_0 \in \mathcal{K}^q_\infty$ and $\left\|e^{t\Delta}u_0\right\|_{\mathcal{K}^q_\infty} \le \|u_0\|_3$ for all $q \in (3, \infty]$.*

Denote $E^q_T := \mathcal{K}^q_T \cap \mathcal{K}^\infty_T$ with $3 < q < \infty$, we have the following lemma.

Lemma 3.1.6. *Let $6 < q < \infty$, $T > 0$. Then the bilinear operator B is bicontinuous from $E^q_T \times E^q_T$ to E^q_T and the following inequality holds*

$$\|B(u,v)\|_{E^q_T} \le C\, \|u\|_{E^q_T}\|v\|_{E^q_T} \ \ \textit{for all } u, v \in E^q_T, \tag{3.4}$$

where C is a positive constant independent of T.

To prove main theorems, we define the auxiliary space G^α, $0 \le \alpha \le 1$ which is made up of the functions $w(x) = (w_1(x), w_2(x), w_3(x)) \in L^3(\mathbb{R}^3)$ such that

$$\sup_{t\ge 0} t^\alpha \left\|e^{t\Delta}|w|\right\|_3 < \infty$$

and

$$\lim_{t\to\infty} t^\alpha \left\|e^{t\Delta}|w|\right\|_3 = 0. \tag{3.5}$$

The norm of the space G^α is defined by

$$\left\|w\right\|_{G^\alpha} := \|w\|_3 + \sup_{t\ge 0} t^\alpha \left\|e^{t\Delta}|w|\right\|_3,$$

where

$$|w(x)| = \Big(\sum_{i=1}^{3} w_i^2(x)\Big)^{1/2}.$$

Lemma 3.1.7. *The space G^α is a Banach space which is invariable with translation in the sense that*

$$\left\|w(\cdot - x_0)\right\|_{G^\alpha} = \left\|w\right\|_{G^\alpha} \ \ \textit{for all} \ \ x_0 \in \mathbb{R}^3.$$

Lemma 3.1.8. *Let $h \in L^\infty$ and $w \in G^\alpha$. Then, $hw \in G^\alpha$ and*

$$\left\|hw\right\|_{G^\alpha} \le \left\|h\right\|_\infty \left\|w\right\|_{G^\alpha}.$$

We define auxiliary the space F^α_T, $0 \le \alpha < 1$, which is made up of the measured functions $u(t,x)$ such that

$$u(t) \in G^\alpha \text{ for all } t \in [0,T] \text{ and } \sup_{0\le t\le T} \left\|u(t)\right\|_{G^\alpha} < \infty.$$

We have the following lemma.

Lemma 3.1.9. *Let $p, \alpha, T \in \mathbb{R}$ be such that*

$$0 \le \alpha \le 1, 3 < q < \infty, T > 0.$$

Then the bilinear operator B is bicontinuous from $E_T^q \times F_T^\alpha$ to F_T^α and from $F_T^\alpha \times E_T^q$ to F_T^α and the following inequalities hold

$$\big\|B(u,v)(t)\big\|_{F_T^\alpha} \lesssim \big\|u\big\|_{E_T^q}\big\|v\big\|_{F_T^\alpha} \text{ for all } u \in E_T^q,\ v \in F_T^\alpha \tag{3.6}$$

and

$$\big\|B(u,v)(t)\big\|_{F_T^\alpha} \lesssim \big\|u\big\|_{F_T^\alpha}\big\|v\big\|_{E_T^q} \text{ for all } u \in F_T^\alpha,\ v \in E_T^q. \tag{3.7}$$

Lemma 3.1.10. *Let $3 \le q < \infty$, $T > 0$, and $0 \le \alpha \le 1$. Then*

(a) *If $u_0 \in L^3$ then $e^{t\Delta}u_0 \in E_T^q$ and $\|e^{t\Delta}u_0\|_{E_T^q} \lesssim \|e^{t\Delta}u_0\|_{K_T^q}$.*

(b) *If $u_0 \in G^\alpha$ then $e^{t\Delta}u_0 \in F_T^\alpha$ and $\big\|e^{t\Delta}u_0\big\|_{F_T^\alpha} \le \|u_0\|_{G^\alpha}$.*

Lemma 3.1.11. *Let $p, \alpha, T \in \mathbb{R}$ be such that*

$$6 < q < 12, 0 \le \alpha \le 1, \text{ and } T > 0.$$

Then there exists a positive constant $C = C(q,\alpha)$ such that for all $u_0 \in G^\alpha$ with $\operatorname{div}(u_0) = 0$ satisfying

$$\sup_{0\le t\le T} t^{\frac{1}{2}(1-\frac{3}{q})}\big\|e^{t\Delta}u_0\big\|_q < C, \tag{3.8}$$

the Navier-Stokes equations (3.1) have a solution $u \in E_T^q \cap F_T^\alpha \cap BC\big([0,T); L^3(\mathbb{R}^3)\big)$. In particular, for arbitrary $u_0 \in L^3$, there exists $T = T(u_0)$ small enough such that the inequality (3.8) holds.

Lemma 3.1.12. *If $u \in C\big([0,\infty); L^3(\mathbb{R}^3)\big)$ is a mild solution of the Navier-Stokes equations (3.1) with the initial value $u_0 \in G^\alpha$, then $u(t) \in G^\alpha$ for all $t > 0$.*

Lemma 3.1.13. *Suppose that $u_0 \in L^{p,r}(\mathbb{R}^3)$ with $1 \le p \le \infty$ and $1 \le r < \infty$. Then $\lim_{n\to\infty}\big\|\mathcal{X}_n u_0\big\|_{L^{p,r}} = 0$, where $\mathcal{X}_n(x) = 0$ for $x \in \{x : \ |x| < n\} \cap \{x : \big|u_0(x)\big| < n\}$, and $\mathcal{X}_n(x) = 1$ otherwise.*

3.2 The decay rates of strong solutions to the Navier-Stokes equations in the three-dimensional space

Consider the Cauchy problem of the incompressible Navier-Stokes equations in the whole space $\mathbb{R}^3$.In this section, we prove that if $u \in C([0,\infty); L^3(\mathbb{R}^3))$ is a strong solution of the Navier-Stokes equations with the initial value u_0 then the time decay rates of u in the L^3-norm coincide with ones of the heat equation with initial value $|u_0|$, see Theorem 2.1.6, in the particular $\alpha = 0$ then we get back the result of I. Gallagher in 2003.

The main result in this section is the decay rates theorem of strong solutions for the Navier-Stokes equations in the three-dimensional space $\mathbb{R}^3$.

Theorem 3.2.1. *Let* $u \in C([0,\infty); L^3(\mathbb{R}^3))$ *be a mild solution of the Navier-Stokes equations* (3.1) *with the initial value* u_0*. Then*

(a) *If* $\big\| e^{t\Delta}|u_0| \big\|_3 = o(t^{-\alpha})$ *with* $0 \le \alpha < 1$*, then* $\big\| u(t) \big\|_3 = o(t^{-\alpha})$.

(b) *If* $\big\| e^{t\Delta}|u_0| \big\|_3 = O(t^{-\alpha})$ *with* $0 \le \alpha \le 1$*, then* $\big\| u(t) \big\|_3 = O(t^{-\alpha})$.

(c) *If* $u_0 \in L^{p,r}(\mathbb{R}^3)$ *with* $1 < p < 3,\ 1 \le r < \infty$*, then* $\big\| e^{t\Delta}|u_0| \big\|_3 = o\big(t^{-\frac{1}{2}\left(\frac{3}{p}-1\right)}\big)$.

(d) *If* $|u_0| \in \dot{B}_3^{-2\alpha,\infty}(\mathbb{R}^3)$ *with* $0 \le \alpha \le 1$*, then* $\big\| e^{t\Delta}|u_0| \big\|_3 = O\big(t^{-\alpha}\big)$.

Conclusions

The thesis has studied the regularity and the decay rates of weak solutions for the Navier-Stokes equations in the unbounded general domain Ω and in the three-dimensional space $\mathbb{R}^3$. Specifically, the thesis has achieved the following three main results:

1. The regularity of weak solutions for the Navier-Stokes equations in a general domain: Suppose u is weak solution of the Navier-Stokes equations in a general domain $\Omega \subseteq \mathbb{R}^3$ and u satisfy the strong energy inequality. Then we can prove that the weak solution u is regular if the kinetic energy $\frac{1}{2}\left\|u(t)\right\|_2^2$ is left-side Hölder continuous with Hölder exponent $\frac{1}{2}$ and with a sufficiently small Hölder semi-norm. The second result in this section proves if $u(t) \in D(A^{)}\frac{1}{4})$ and $\varliminf\limits_{\delta \to 0^+} \left\|A^{\frac{1}{4}}\big(u(t-\delta) - u(t)\big)\right\|_2 < C$ for all $t \in [0, T)$ and for C being a small enough positive constant then u is regular in $[0, T)$. This result has been published in the article [1] in the List of published scientific publications related to the thesis.

2. The decay rates of weak solutions to the Navier-Stokes equations in the general domain: We proved that the weak solution u of the Navier-Stokes equations with the norm in $L^2(\Omega)$ has the same decay rates in time as the solution of a homogeneous Stokes system with the same initial value and the exponent is less than $\dfrac{3}{4}$. Moreover, when adding some conditions of the initial value, the weak solution u coincides with the solution of the homogeneous Stokes system as the time t goes to infinity. This result has been published in the article [2] in the List of published scientific publications related to the thesis.

3. The decay rates of strong solutions to the Navier-Stokes equations in the three-dimensional space: We proved that the decay rates of strong solutions to the Cauchy problem of the Navier-Stokes equations. Let $u \in C([0, \infty); L^3(\mathbb{R}^3))$ be a strong solution of the Cauchy problem for the Navier-Stokes equations with the initial value u_0 in the three-dimensional space. We prove that the time decay rates of u in the L^3-norm coincide with ones of the heat equation with the initial value $|u_0|$. This result has been published in the article [3] in the List of published scientific publications related to the thesis.

List of the scientific publications related to the thesis

[1] Duong V. T. T., Khai D. Q., Tri N. M. (2020), " On regularity of weak solutions for the Navier-Stokes equations in general domains ", *Mathematische Nachrichten*, accepted.

[2] Duong V. T. T., Khai D. Q. (2020), "L^2 Decay of weak solutions for the Navier-Stokes equations in general domains ", *Journal of Science and Technology Thai Nguyen University*, 225(02), pp. 45-51.

[3] Duong V.T.T., Khai D.Q., Tri N.M. (2020), "Time decay rates of the L^3-norm for strong solutions to the Navier-Stokes equations in $\mathbb{R}^3$ ", *Journal of Mathematical Analysis and Applications*, 485(2), pp. 81-98.